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Algebraic structure of lepton and quark flavor invariants and CP violation

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Abstract: Lepton and quark flavor invariants are studied, both in the Standard Model with a dimension five Majorana neutrino mass operator, and in the seesaw model. The ring of invariants in the lepton sector is highly non-trivial, with non-linear relations among the basic invariants. The invariants are classified for the Standard Model with two and three generations, and for the seesaw model with two generations, and the Hilbert series is computed. The seesaw model with three generations proved computationally too difficult for a complete solution. We give an invariant definition of the CP-violating angle $\bar{\vartheta}$ in the electroweak sector.

Keywords: Quark Masses and SM Parameters, Neutrino Physics, Global Symmetries, Standard Model

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Contents

1 Introduction

Flavor violation of quarks and leptons by Standard Model weak interactions is parameterized by unitary 3×3 matrices, the CKM matrix in the quark sector and the PMNS matrix in the lepton sector. The fundamental parameters in the Standard Model are the quark and charged lepton Yukawa coupling matrices and the flavor matrix of the dimension-five Majorana mass operator for neutrinos [1]. The fermion masses and mixing angles are derived quantities, obtained from the eigenvalues and eigenvectors of the flavor matrices in the low-energy theory. In the mass eigenstate basis, one still has the freedom to make phase rotations on the fermions fields, which leads to the redefinition of the CKM matrix

$$V \to e^{-i\Phi_U} V e^{i\Phi_D} \tag{1.1}$$

in the quark sector, where $\Phi_U = \mathrm{diag}(\phi_u, \phi_c, \phi_t)$ and $\Phi_D = \mathrm{diag}(\phi_d, \phi_s, \phi_b)$. Physical quantities are basis independent, and must be invariant under the rephasing eq. (1.1). CKM rephasing invariants have been studied extensively in the literature [2–5], the best-known example being the CP-odd Jarlskog invariant $J = \mathrm{Im}\, V_{11} V_{22} V^*_{12} V^*_{21}$. Rephasing invariance also exists for the lepton mixing matrix. In a previous paper [5], we extended the analysis of rephasing invariants to give a complete classification of these invariants for the Standard Model, and for the seesaw model [6].

The parameterization of the flavor structure in terms of masses and mixing angles is convenient for computing decay rates and scattering amplitudes. However, if one wants to understand the origin of flavor structure, the more fundamental quantities are the flavor matrices in the Lagrangian from which the masses and mixing angles are derived by diagonalization. A well-known difficulty is that the flavor matrices are basis-dependent, since one can make unitary transformations on the quark and lepton fields in the Lagrangian. For example, the Yukawa matrix for charge 2/3 quarks transforms as

$$Y_U \to \mathrm{U}_{U^c}{}^T \; Y_U \; \mathrm{U}_Q \tag{1.2}$$

where U_Q and U_{U^c} are unitary transformations on the quark doublet and singlet fields. Observable quantities must be independent of this change of basis, i.e. invariant under eq. (1.2), and such quantities are sometimes referred to as weak basis invariants [7, 8]. One can check the predictions of a flavor model by comparing invariant quantities with their corresponding experimental values.

Classifying invariants also is important in analyzing theories which explain flavor by a dynamical mechanism. Examples of this type were studied in the early literature on unified theories [9, 10] in the context of understanding gauge symmetry breaking patterns by minimizing Higgs potentials. A recent example from flavor physics needing the classification of invariants can be found in ref. [11].

There is an extensive literature on quark and lepton invariants (see, e.g. [7, 8, 12–15]). The main emphasis in previous work has been the study of CP violation. CP-violating invariants analogous to the Jarlskog invariant were written down. The vanishing of the CP-violating invariants was sufficient to guarantee the vanishing of CP violation in the CKM and PMNS mixing matrices.

In this paper, we take a different approach, studying all the invariants, and treating the problem using the methods of invariant theory [16–18], which considers the ring of polynomials that are invariant under the action of a group. Polynomial invariants also are the relevant objects for physics applications, since an effective Lagrangian is written as a polynomial in the basic variables which describe the theory.[1] A basic result of invariant theory is that the ring of invariants has a finite number of generators. There can be

[1]For example, the chiral Lagrangian is a polynomial in the quark mass matrix M.

non-trivial relations among the invariants, known as syzygies [19], so that the invariant ring need not be a free ring. The number of invariants of a given degree is encoded in the Hilbert series. The complete classification of the invariant ring is, in general, a very difficult computational problem.

We study the invariants of the Standard Model low-energy theory and the seesaw theory in both the quark and lepton sectors. In the quark sector, the complete structure of the invariant ring is given, and the relation between the polynomial invariants and rephasing invariants also is given. The structure of the invariant ring in the lepton sector is considerably more involved than in the quark sector. The classification of lepton invariants is given for the low-energy effective Standard Model theory for two and three generations. For the high-energy seesaw theory, the classification is given for two generations. For three generations, we have been unable to completely classify all the relations or to determine the Hilbert series because the problem is computationally too difficult. The simpler invariants (i.e. of small degree) are given for this case.

The paper is organized as follows. Section 2 defines the high-energy seesaw theory and its low-energy effective theory. The flavor-symmetry breaking matrices and ϑ-angles of each theory are given, together with their transformation properties under flavor symmetry and CP. Section 3 defines the mass and mixing matrices of the high-energy and low-energy theories, explains the counting of mixing angles and phases, and discusses rephasing invariance. Section 4 provides a brief introduction to the mathematics of invariant theory that we need for our analysis. Section 5 reviews the classification of the quark mass matrix invariants. Sections 6 and 7 consider the classification of lepton mass matrix invariants for two and three generations of fermions, respectively, in both the low-energy effective theory and the seesaw theory.

2 Flavor symmetries

We consider the SU(3) $\times$ SU(2) $\times$ U(1) gauge theory with n_g generations of Standard Model fermions and n_g' generations of gauge singlet fermions (neutrino singlets). The fermion multiplets are $Q_i = (\mathbf{3},\mathbf{2})_{1/6}$, $U_i^c = (\bar{\mathbf{3}},\mathbf{1})_{-2/3}$, $D_i^c = (\bar{\mathbf{3}},\mathbf{1})_{1/3}$, $L_i = (\mathbf{1},\mathbf{2})_{-1/2}$ and $E_i^c = (\mathbf{1},\mathbf{1})_1$, $i = 1,\ldots,n_g$, and $N_I^c = (\mathbf{1},\mathbf{1})_0$, $I = 1,\ldots,n_g'$. All fermion multiplets are left-handed Weyl fields. The fermion multiplets with $n_g' = n_g$ have a natural embedding in the **16** spinor representation of SO(10).

The flavor symmetry of the fermion sector of the high-energy theory is $\mathrm{SU}(n_g)^5 \times \mathrm{U}(n_g') \times \mathrm{U}(1)^2$, since there is a separate $\mathrm{SU}(n_g)$ flavor symmetry for each of the five multiplets Q, U^c, D^c, L and E^c, a $\mathrm{U}(n_g')$ flavor symmetry for the singlets N^c, and two additional non-anomalous U(1) flavor symmetries. Out of the six possible U(1) symmetries, only three linear combinations are non-anomalous under SU(3) $\times$ SU(2) $\times$ U(1): N^c number which is included in $\mathrm{U}(n_g')$, $(B-L)$, and $(E^c + D^c - U^c)$ number. The three additional anomalous U(1) groups can be treated as symmetries if the three ϑ-angles[2] $\vartheta_{3,2,1}$ of the SU(3), SU(2)

[2]The ϑ angles multiplying $F\tilde{F}$ terms are not to be confused with angles θ of the quark and lepton mixing matrices. There are no instantons in the U(1) sector, but the ϑ angle can have physical consquences in the presence of topological defects.

and U(1) gauge groups transform under arbitrary chiral phase transformations $\psi \to e^{i\alpha_\psi}\psi$ on the fields $\psi = Q$, U^c, D^c, L and E^c as

$$\begin{aligned}\vartheta_3 &\to \vartheta_3 - n_g\left(2\alpha_Q + \alpha_{U^c} + \alpha_{D^c}\right),\\ \vartheta_2 &\to \vartheta_2 - n_g\left(3\alpha_Q + \alpha_L\right),\\ \vartheta_1 &\to \vartheta_1 - n_g\left(\frac{1}{6}\alpha_Q + \frac{4}{3}\alpha_{U^c} + \frac{1}{3}\alpha_{D^c} + \frac{1}{2}\alpha_L + \alpha_{E^c}\right).\end{aligned} \tag{2.1}$$

Eq. (2.1) does not depend on n'_g or α_{N^c}, since N^c are gauge singlets. With the transformation eq. (2.1), the chiral flavor symmetry becomes $\mathrm{U}(n_g)^5 \times \mathrm{U}(n'_g)$, with a separate flavor factor for each of the six fermion multiplets.

The $\mathrm{U}(n_g)^5 \times \mathrm{U}(n'_g)$ flavor symmetry of the fermion and gauge kinetic energy terms is explicitly broken by gauge-invariant renormalizable terms — Yukawa couplings between fermion multiplets and the Higgs doublet and Majorana mass terms of the fermion singlets. The flavor symmetry-breaking Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-U^c_i\left(Y_U\right)_{ij} Q_j H - D^c_i\left(Y_D\right)_{ij} Q_j H^\dagger\\ &-E^c_i\left(Y_E\right)_{ij} L_j H^\dagger - N^c_I\left(Y_\nu\right)_{Ij} L_j H\\ &-\frac{1}{2}N^c_I M_{IJ} N^c_J + \text{h.c.},\end{aligned} \tag{2.2}$$

where $H = (1,2)_{1/2}$ is the Higgs doublet, and gauge and Lorentz indices have been suppressed. The Yukawa couplings $Y_{U,D,E}$ are $n_g \times n_g$ matrices, whereas the neutrino Yukawa coupling Y_ν is an $n'_g \times n_g$ matrix. The singlet neutrino Majorana mass matrix M is a symmetric $n'_g \times n'_g$ matrix. In the Standard Model without neutrino singlets, renormalizable terms proportional to Y_ν and M are absent.

Under the chiral flavor symmetry transformations $\psi \to \mathcal{U}_\psi\, \psi$, where $\mathcal{U}_\psi$ are unitary matrices in flavor space for the fermion fields $\psi = Q$, U^c, D^c, L, E^c and N^c, the Yukawa coupling matrices, the Majorana mass matrix and the ϑ angles transform as

$$\begin{aligned}Y_U &\to {\mathcal{U}_{U^c}}^T\, Y_U\, \mathcal{U}_Q,\\ Y_D &\to {\mathcal{U}_{D^c}}^T\, Y_D\, \mathcal{U}_Q,\\ Y_E &\to {\mathcal{U}_{E^c}}^T\, Y_E\, \mathcal{U}_L,\\ Y_\nu &\to {\mathcal{U}_{N^c}}^T\, Y_\nu\, \mathcal{U}_L,\\ M &\to {\mathcal{U}_{N^c}}^T\, M\, \mathcal{U}_{N^c},\\ \vartheta_3 &\to \vartheta_3 - 2\arg\det\mathcal{U}_Q - \arg\det\mathcal{U}_{U^c} - \arg\det\mathcal{U}_{D^c},\\ \vartheta_2 &\to \vartheta_2 - 3\arg\det\mathcal{U}_Q - \arg\det\mathcal{U}_L,\\ \vartheta_1 &\to \vartheta_1 - \frac{1}{6}\arg\det\mathcal{U}_Q - \frac{4}{3}\arg\det\mathcal{U}_{U^c} - \frac{1}{3}\arg\det\mathcal{U}_{D^c} - \frac{1}{2}\arg\det\mathcal{U}_L - \arg\det\mathcal{U}_{E^c}.\end{aligned} \tag{2.3}$$

Under CP, each matrix is transformed to its complex conjugate, and each ϑ angle changes sign,

$$\begin{aligned}Y_{U,D,E,\nu} &\to Y^*_{U,D,E,\nu}\,,\\ M &\to M^*\,,\\ \vartheta_{1,2,3} &\to -\vartheta_{1,2,3}\,.\end{aligned} \tag{2.4}$$

Under the chiral flavor symmetry transformation, the ϑ angles are shifted by eq. (2.3). The invariant angle $\bar{\vartheta}_{\rm QCD}$ is defined by

$$\bar{\vartheta}_{\rm QCD} = \vartheta_3 + \arg\det Y_U + \arg\det Y_D\,. \tag{2.5}$$

The analogous angles $\bar{\vartheta}_{1,2}$ can not be separately defined, but one can define an invariant ϑ-parameter in the electroweak sector

$$\bar{\vartheta}_{\rm EW} = \vartheta_2 + 2\vartheta_1 + \frac{8}{3}\arg\det Y_U + \frac{2}{3}\arg\det Y_D + 2\arg\det Y_E\,. \tag{2.6}$$

After electroweak symmetry breaking, the QED ϑ-angle is $2\bar{\vartheta}_{\rm QED} = \bar{\vartheta}_{\rm EW}$. The factor of two arises because the generators for a non-abelian gauge theory are normalized to $\mathrm{Tr}\, T^a T^b = \delta^{ab}/2$.

In the absence of electroweak symmetry breaking, there are n_g' massive Majorana neutrino singlets with masses of $\mathcal{O}(M)$, the heavy Majorana neutrino mass scale, and all other fermions are strictly massless. It is natural that M be of order the GUT scale, the scale at which the GUT gauge symmetry breaks to the Standard Model gauge group, under which the N^c fields are uncharged. When the Higgs field gets a vacuum expectation value $v/\sqrt{2}$, the Yukawa matrices generate Dirac mass matrices for the quarks and leptons,

$$m_{U,D,E,\nu} = Y_{U,D,E,\nu}\,\frac{v}{\sqrt{2}}\,, \tag{2.7}$$

with the same flavor transformation properties as the Yukawa couplings. The Dirac and Majorana mass matrices of the $(n_g + n_g')$ left-handed neutrino fields combine to form a neutrino mass term

$$-\frac{1}{2}\mathcal{N}_{\mathcal{I}}\,(M_{\mathcal{N}})_{\mathcal{I}\mathcal{J}}\,\mathcal{N}_{\mathcal{J}}, \qquad 1 \le \mathcal{I},\mathcal{J} \le n_g + n_g' \tag{2.8}$$

where the $(n_g+n_g')\times(n_g+n_g')$ neutrino mass matrix $M_{\mathcal{N}}$ is equal to the symmetric matrix

$$M_{\mathcal{N}} \equiv \begin{pmatrix} 0 & {m_\nu}^T \\ m_\nu & M \end{pmatrix}\,. \tag{2.9}$$

The (n_g+n_g') neutrino fields $\mathcal{N}_{\mathcal{I}}$ are (ν_i, N_I^c). The (n_g+n_g') mass eigenstates of eq. (2.9) give the Majorana mass-eigenstate neutrino fields, which are linear combinations of ν_i and N_I^c. The heavy neutrinos with masses $\mathcal{O}(M)$ are predominantly N^c with an $\mathcal{O}(v/M)$ admixture of ν, and the light neutrinos with masses $\mathcal{O}(v^2/M)$ are predominantly ν with an $\mathcal{O}(v/M)$ admixture of N^c.

A low-energy effective field theory can be obtained from the seesaw theory by integrating out the n_g' heavy Majorana neutrino mass eigenstates. In this low-energy theory, the leading flavor symmetry-breaking Lagrangian is given by

$$\mathcal{L}^{\rm EFT} = -U_i^c\,(Y_U)_{ij}\,Q_j H - D_i^c\,(Y_D)_{ij}\,Q_j H^\dagger - E_i^c\,(Y_E)_{ij}\,L_j H^\dagger + \frac{1}{2}(L_i H)\,(C_5)_{ij}\,(L_j H) + \text{h.c.}, \tag{2.10}$$

where the coefficient of the dimension-five operator [1] is given by

$$C_5 = Y_\nu^T\, M^{-1}\, Y_\nu \tag{2.11}$$

to lowest order in the $1/M$ expansion. When the electroweak gauge symmetry breaks, the dimension-five operator yields an effective $n_g \times n_g$ Majorana mass matrix

$$m_5 = -C_5 v^2/2 \tag{2.12}$$

for the (primarily) weak doublet neutrinos. Under the flavor symmetries and CP, the flavor matrices $Y_{U,D,E}$ and ϑ angles $\vartheta_{1,2,3}$ of the low-energy effective theory transform under chiral flavor symmetry and CP as in eq. (2.3) and eq. (2.4), respectively, whereas C_5 transforms as

$$\begin{aligned} C_5 &\to {\mathcal{U}_L}^T\, C_5\, \mathcal{U}_L, \\ C_5 &\to C_5^*\,, \end{aligned} \tag{2.13}$$

respectively.

We will analyze the flavor structure of both the seesaw theory and its low-energy effective theory. The analysis depends only on the flavor transformation properties of the Yukawa coupling and Majorana mass matrices (i.e. the fermion mass matrices). Thus, it applies to any theory which has Dirac and Majorana mass matrices with the same transformation properties as given here, regardless of whether the Dirac mass terms are proportional to Yukawa couplings in the theory, or are generated by some mechanism from more fundamental parameters of the theory.

3 Masses, mixing angles and phases

In this section, we define the mass and mixing parameters of the high-energy seesaw theory and its low-energy effective theory. Most of the section is a review of well-known results, and serves to define the parameters and notation which are needed later. The mass matrices of the high and low energy theories in the weak eigenstate basis are transformed to the mass eigenstate basis by flavor rotations to obtain the fermion masses and mixing matrices. The counting of mixing angles and phases for the case $n_g' = n_g$ follows the analysis of ref. [5]. The counting of physical parameters is given here for the cases $n_g' > n_g$ and $n_g' < n_g$, for completeness. An alternative way of counting parameters, analogous to the counting of Goldstone bosons, is given in refs. [20, 21].

Any complex matrix M can be written in the form $M = \mathsf{U}\, \Lambda\, \mathsf{U}'$ where U and U' are unitary matrices, and Λ is a diagonal matrix with real, non-negative entries. If M is also a symmetric matrix, then it can be written in the form $M = M^T = \mathsf{U}^T\, \Lambda\, \mathsf{U}$, where U is a unitary matrix.

3.1 High-energy theory

The flavor matrices of the high-energy seesaw theory are written in eq. (2.2) in the weak eigenstate basis. These flavor matrices are related to the mass eigenstate basis by

$$\begin{aligned} Y_U &= \mathsf{U}_{U^c}\ \Lambda_U\ \mathsf{U}_U, \\ Y_D &= \mathsf{U}_{D^c}\ \Lambda_D\ \mathsf{U}_D, \\ Y_E &= \mathsf{U}_{E^c}\ \Lambda_E\ \mathsf{U}_E, \\ Y_\nu &= \mathsf{U}_{N^c}\ \Lambda_\nu\ \mathsf{U}_\nu, \\ M &= \mathsf{U}'^{\,T}_{N^c}\ \Lambda_N\ \mathsf{U}'_{N^c}, \end{aligned} \tag{3.1}$$

where $\Lambda_{U,D,E}$, Λ_ν and Λ_N are $n_g \times n_g$, $n_g' \times n_g$ and $n_g' \times n_g'$ diagonal matrices respectively, with real, non-negative entries; U_{U^c,D^c,E^c} and $\mathsf{U}_{U,D,E,\nu}$ are $n_g \times n_g$ unitary matrices, and U_{N^c} and U'_{N^c} are $n_g' \times n_g'$ unitary matrices, which transform the mass eigenstate basis to the weak eigenstate basis. Performing the chiral flavor transformation eq. (2.3) with ${\mathcal{U}_{U^c}}^T = {\mathsf{U}_U}^{-1}$, ${\mathcal{U}_{D^c}}^T = {\mathsf{U}_D}^{-1}$, ${\mathcal{U}_{E^c}}^T = {\mathsf{U}_E}^{-1}$, $\mathcal{U}_Q = {\mathsf{U}_U}^{-1}$, $\mathcal{U}_L = {\mathsf{U}_E}^{-1}$, and $\mathcal{U}_{N^c} = {\mathsf{U}'_{N^c}}^{-1}$ brings the flavor matrices to the form

$$\begin{aligned} Y_U &= \Lambda_U, \\ Y_D &= \Lambda_D\ V_{\rm CKM}^{-1}, \\ Y_E &= \Lambda_E, \\ Y_\nu &= W^{-1}\ \Lambda_\nu\ V, \\ M &= \Lambda_N, \end{aligned} \tag{3.2}$$

where $V_{\rm CKM} \equiv \mathsf{U}_U {\mathsf{U}_D}^{-1}$, $V \equiv \mathsf{U}_\nu {\mathsf{U}_E}^{-1}$ and $W \equiv {\mathsf{U}_{N^c}}^{-1} (\mathsf{U}'_{N^c})^T$ are the three unitary matrices which describe flavor mixing in the seesaw theory. $V_{\rm CKM}$ is the Cabibbo-Kobayashi-Maskawa mixing matrix in the quark sector. As is well-known, this $n_g \times n_g$ matrix corresponds to the mismatch between the unitary field redefinitions on U and D in the quark doublets Q required to diagonalize Y_U and Y_D. V is the analogue of the CKM matrix in the lepton sector; it is the $n_g \times n_g$ matrix corresponding to the mismatch between the unitary field redefinitions on ν and E in the lepton doublets L required to diagonalize Y_ν and Y_E. W is an $n_g' \times n_g'$ mixing matrix in the lepton sector corresponding to the mismatch between the unitary field redefinitions on N^c required to diagonalize M and Y_ν.

To proceed further, it is necessary to consider the three cases $n_g' = n_g$, $n_g' < n_g$ and $n_g' > n_g$ individually. We first specialize to the case $n_g' = n_g$ considered previously in ref. [5] and review the analysis given there. The analysis is then generalized to the cases $n_g' \neq n_g$. The quark sector only depends on the number of quark generations n_g, but the lepton sector analysis depends on whether $n_g' = n_g$, $n_g' < n_g$ or $n_g' > n_g$.

3.1.1 $n_g' = n_g$

The real diagonal matrices $\Lambda_{U,D,E,\nu,N}$ are invariant under the rephasings,

$$\begin{aligned} \Lambda_\psi &\to e^{-i\Phi_\psi}\ \Lambda_\psi\ e^{i\Phi_\psi},\ \ \psi = U, D, E, \\ \Lambda_\nu &\to e^{-i\Phi_\nu}\ \Lambda_\nu\ e^{i\Phi_\nu}, \\ \Lambda_N &\to \eta_N\ \Lambda_N\ \eta_N, \end{aligned} \tag{3.3}$$

Matrices	Masses	Angles	Phases
Λ_U	n_g	0	0
Λ_D	n_g	0	0
$V_{\rm CKM}$	0	$\frac{1}{2}n_g(n_g-1)$	$\frac{1}{2}(n_g-1)(n_g-2)$
Total	$2n_g$	$\frac{1}{2}n_g(n_g-1)$	$\frac{1}{2}(n_g-1)(n_g-2)$

Table 1. Parameters in the quark sector for n_g generations. The Λ_U and Λ_D rows give the parameters if Y_U or Y_D are considered separately, and the third row gives the *additional* parameters if both Y_U and Y_D are considered together. There are $(n_g-1)^2$ mixing parameters (angles plus phases), and a total of $({n_g}^2+1)$ parameters.

where $\Phi_{U,D,E,\nu}$ are real diagonal matrices, and η_N is a diagonal matrix with allowed eigenvalues ±1. Only ±1 rephasings are allowed for the Majorana fields N^c. Under these rephasings, the mixing matrices $V_{\rm CKM}$, V and W transform as

$$\begin{aligned} V_{\rm CKM} &\to e^{-i\Phi_U}\ V_{\rm CKM}\ e^{i\Phi_D}, \\ V &\to e^{-i\Phi_\nu}\ V\ e^{i\Phi_E}, \\ W &\to e^{-i\Phi_\nu}\ W\ \eta_N\,. \end{aligned} \tag{3.4}$$

Quark sector. The parameter counting in the quark sector is well-known, and is summarized here for completeness. The matrices Λ_U and Λ_D each contain n_g eigenvalues, which correspond to the U-quark and D-quark masses, respectively, and are CP even. The quark mixing matrix $V_{\rm CKM}$ is an $n_g \times n_g$ unitary matrix with ${n_g}^2$ parameters. It is conventional to divide these parameters into angles and phases — angles are even under CP, whereas phases are odd under CP. If the $V_{\rm CKM}$ matrix is CP invariant, it is an $n_g \times n_g$ real orthogonal matrix with $n_g(n_g-1)/2$ parameters. The unitary matrix $V_{\rm CKM}$ has $n_g(n_g-1)/2$ angles and $n_g(n_g+1)/2$ phases, and can be parametrized by

$$e^{i\chi}\ e^{i\Phi}\ \mathcal{V}(\theta_i,\delta_i)\ e^{i\Psi}\,, \tag{3.5}$$

where χ is an overall phase, $\Phi = \mathrm{diag}(0,\phi_2,\ldots,\phi_{n_g})$, and $\Psi = \mathrm{diag}(0,\psi_2,\ldots,\psi_{n_g})$. The phase redefinitions Φ_U and Φ_D of $V_{\rm CKM}$ in eq. (3.4) can be chosen to remove the $2n_g-1$ phases χ, ϕ_i, ψ_i, $i=2,\ldots,n_g$.[3] Thus, $V_{\rm CKM}$ has $n_g(n_g+1)/2-(2n_g-1)=(n_g-1)(n_g-2)/2$ net phases. This counting of parameters is summarized in table 1.

We choose a parameterization $V_{\rm CKM} = \mathcal{V}(\theta_i,\delta_i)$ in terms of a standard functional form $\mathcal{V}$, where the $n_g(n_g-1)/2$ angles $\theta_i \in [0,\pi/2]$ and the $(n_g-1)(n_g-2)/2$ phases $\delta_i \in [0,2\pi)$. The CKM matrix for $n_g=3$ is given by [22]

$$\mathcal{V}(\theta_{12},\theta_{13},\theta_{23},\delta) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.6}$$

where $s_i \equiv \sin\theta_i$ and $c_i \equiv \cos\theta_i$. It is now conventional to call the angles $\theta_{23},\theta_{13},\theta_{12}$ rather than $\theta_1,\theta_2,\theta_3$. The standard form eq. (3.6) has $\det\mathcal{V}=1$.

[3] There are n_g phases each in Φ_U and Φ_D, but the transformation $\Phi_U=\Phi_D \propto \mathbb{1}$ leaves $V_{\rm CKM}$ invariant.

Lepton sector. The matrices Λ_N and Λ_E each have n_g eigenvalues which are CP even. The lepton mixing matrices V and W are $n_g \times n_g$ unitary matrices, which can be parametrized by

$$\begin{aligned} V &= e^{i\chi}\, e^{i\Phi}\, \mathcal{V}(\theta_i, \delta_i)\, e^{i\Psi/2}, \\ W &= e^{i\chi'}\, e^{i\Phi'}\, \mathcal{V}(\theta_i', \delta_i')\, e^{i\Psi'/2}. \end{aligned} \tag{3.7}$$

We use the same standard functional form $\mathcal{V}$ as for the quark sector, but with different numerical values for the arguments θ_i and δ_i.[4] The factor of two in Ψ and Ψ' will be explained below.

The rephasing transformations Φ_ν, Φ_E and η_N of eq. (3.4) can be used to (i) eliminate χ, χ' and ψ_i, (ii) restrict ψ_i' to the range $[0, 2\pi)$ rather than $[0, 4\pi)$, and (iii) eliminate *either* Φ *or* Φ', *but not both.* It is convenient to use the same domain $[0, 2\pi)$ for all phases, which is why Ψ' was scaled by a factor of 2.

First consider amplitudes which depend only on Y_ν and Y_E, but not on M. In this case, the mixing matrix W is no longer observable and can be set to unity. The mixing matrix V has $(2n_g - 1)$ allowed phase redefinitions: n from Φ_ν, n from Φ_E, and minus one, because $\Phi_\nu = \Phi_E \propto \mathbb{1}$ does not change V. Thus, the parameter counting for the mixing matrix V is identical to that for $V_{\rm CKM}$ in the quark sector, with $n_g(n_g - 1)/2$ angles, and $(n_g - 1)(n_g - 2)/2$ phases. Similarly, for amplitudes depending only on M and Y_ν and not on Y_E, the mixing matrix V is no longer observable and can be set to unity. The mixing matrix W has n_g allowed phase redefinitions Φ_ν. Thus, there are $n_g(n_g - 1)/2$ angles and $n_g(n_g + 1)/2 - n_g = n_g(n_g - 1)/2$ phases. If the three matrices M, Y_ν and Y_E are considered together, then the mixing matrices V and W together can have $2n_g$ allowed phase redefinitions due to Φ_ν and Φ_E. As compared with the case of only V or only W, where there were $2n_g - 1 + n_g$ phase redefinitions possible, we have $(n_g - 1)$ fewer phase redefinitions, and hence $(n_g - 1)$ additional observable phases. These $(n_g - 1)$ additional phases occur because the same phase redefinition Φ_ν was present for both V and W, and so cannot be chosen to remove phases from both V and W. Thus, there are an additional $(n_g - 1)$ phases if all three mass matrices are considered together. These phases can be included in either V or W. The standard form of the mixing matrices which uses the Φ_ν phases to eliminate the Φ phases from V is given by

$$\begin{aligned} V &= \mathcal{V}(\theta_i, \delta_i), \\ W &= e^{-i\bar{\Phi}}\, \mathcal{V}(\theta_i', \delta_i')\, e^{i\Psi'/2}, \end{aligned} \tag{3.8}$$

whereas the standard form of the mixing matrices which uses the Φ_ν phases to eliminate the Φ' phases from W is given by

$$\begin{aligned} V &= e^{i\bar{\Phi}}\, \mathcal{V}(\theta_i, \delta_i), \\ W &= \mathcal{V}(\theta_i', \delta_i')\, e^{i\Psi'/2}. \end{aligned} \tag{3.9}$$

[4]The use of the same symbols θ_i for the quark and lepton sectors should cause no confusion, since we do not need to deal with mixing in both sectors simultaneously.

Matrices	Masses	Angles	Phases
Λ_N	n_g	0	0
Λ_ν	n_g	0	0
Λ_E	n_g	0	0
$V : Y_\nu, Y_E$	0	$\frac{1}{2} n_g(n_g-1)$	$\frac{1}{2}(n_g-1)(n_g-2)$
$W : M, Y_\nu$	0	$\frac{1}{2} n_g(n_g-1)$	$\frac{1}{2} n_g(n_g-1)$
$\bar{\Phi} \not\propto \mathbb{1}$	0	0	n_g-1
Total	$3n_g$	$n_g(n_g-1)$	$n_g(n_g-1)$

Table 2. Parameters in the lepton sector for $n'_g = n_g$ generations. The Λ_N, Λ_ν and Λ_E rows give the parameters if M or Y_ν or Y_E are considered separately. The V and W rows give the *additional* parameters if both Y_ν and Y_E, or both M and Y_ν are considered together, respectively. The last row gives the *additional* parameters to those in the previous rows when all three matrices M, Y_ν and Y_E are considered together. There are $2n_g(n_g-1)$ mixing parameters (angles and phases), and a total of $n_g(2n_g+1)$ parameters.

Matrices	Masses	Angles	Phases
Λ_N	n'_g	0	0
Λ_ν	n'_g	0	0
Λ_E	n_g	0	0
$V : Y_\nu, Y_E$	0	$\frac{1}{2} n_g(n_g-1)$	$\frac{1}{2} n_g(n_g-1) - n'_g + 1$
$W : M, Y_\nu$	0	$\frac{1}{2} n'_g(n'_g-1)$	$\frac{1}{2} n'_g(n'_g-1)$
$\bar{\Phi} \not\propto \mathbb{1}$	0	0	n'_g-1
$U_{n_g-n'_g}$	0	$\frac{1}{2}(n_g-n'_g)(n_g-n'_g-1)$	$\frac{1}{2}(n_g-n'_g)(n_g-n'_g+1)$
Total	$n_g+2n'_g$	$n_g n'_g - n'_g$	$n_g n'_g - n_g$

Table 3. Parameters in the lepton sector for n_g fermion generations and $n'_g < n_g$ neutrino singlets. The total number of parameters is equal to the sum of the first six rows minus the last row. The parameters in $U_{n_g-n'_g}$ are removed from V.

In eq. (3.8), V has the canonical CKM form with $n_g(n_g-1)/2$ angles θ_i and $(n_g-1)(n_g-2)/2$ phases δ_i, whereas in eq. (3.9), W has the canonical PMNS form with $n_g(n_g-1)/2$ angles θ'_i and $n_g(n_g-1)/2$ phases consisting of the $(n_g-1)(n_g-2)/2$ phases δ_i and the (n_g-1) phases ψ'_i. In either basis, there are (n_g-1) additional phases $\bar{\Phi} \equiv \Phi - \Phi'$ which cannot be removed, and are observable. This parameter counting for $n'_g = n_g$ is summarized in table 2.

ϑ angles. Once the mixing matrices have been put in standard form, one can perform additional phase rotations which leave the mixing matrices invariant to eliminate ϑ angles. The only allowed transformation is an overall phase rotation with $\Phi_U = \Phi_D = \phi_Q \mathbb{1}$, i.e.

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Matrices	Masses	Angles	Phases
Λ_N	n'_g	0	0
Λ_ν	n_g	0	0
Λ_E	n_g	0	0
$V : Y_\nu, Y_E$	0	$\frac{1}{2}n_g(n_g-1)$	$\frac{1}{2}(n_g-1)(n_g-2)$
$W : M, Y_\nu$	0	$\frac{1}{2}n'_g(n'_g-1)$	$\frac{1}{2}n'_g(n'_g+1)-n_g$
$\bar{\Phi} \not\propto \mathbb{1}$	0	0	$n_g - 1$
$U_{n'_g-n_g}$	0	$\frac{1}{2}(n'_g-n_g)(n'_g-n_g-1)$	$\frac{1}{2}(n'_g-n_g)(n'_g-n_g+1)$
Total	$2n_g+n'_g$	$n_g n'_g - n_g$	$n_g n'_g - n_g$

Table 4. Parameters in the lepton sector for n_g fermion generations and $n'_g > n_g$ neutrino singlets. The total number of parameters is equal to the sum of the first six rows minus the last row. The parameters in $U_{n'_g-n_g}$ are removed from W.

baryon number. Under this phase transformation,

$$\begin{aligned} \vartheta_3 &\to \vartheta_3, \\ \vartheta_2 &\to \vartheta_2 - 3n_g\phi_Q, \\ \vartheta_1 &\to \vartheta_1 + \frac{3}{2}n_g\phi_Q \ . \end{aligned} \tag{3.10}$$

The transformation leaves ϑ_3 and $\vartheta_2 + 2\vartheta_1$ unchanged, so there are two physical ϑ angles remaining: $\overline{\vartheta}_{\rm QCD}$, the strong interaction CP-angle in the basis where the quark mass matrices are real and diagonal, and $\overline{\vartheta}_{EW} = \vartheta_2 + 2\vartheta_1$, the electroweak CP-angle in the basis where the quark and charged lepton mass matrices are real and diagonal.

3.1.2 $n'_g < n_g$

For $n'_g < n_g$, the $n'_g \times n_g$ diagonal matrix Λ_ν can be written as

$$\Lambda_\nu \equiv \begin{bmatrix} \bar{\Lambda}_\nu & 0 \end{bmatrix}, \tag{3.11}$$

where 0 denotes the $n'_g \times (n_g - n'_g)$ zero matrix, and $\bar{\Lambda}_\nu$ is a diagonal $n'_g \times n'_g$ matrix with n'_g real non-negative eigenvalues. This matrix is invariant under

$$\begin{bmatrix} \bar{\Lambda}_\nu & 0 \end{bmatrix} \to e^{-i\Phi_\nu} \begin{bmatrix} \bar{\Lambda}_\nu & 0 \end{bmatrix} \begin{bmatrix} e^{i\Phi_\nu} & 0 \\ 0 & U_{n_g-n'_g} \end{bmatrix}, \tag{3.12}$$

where $U_{n_g-n'_g}$ denotes an arbitrary $(n_g - n'_g) \times (n_g - n'_g)$ unitary matrix. The rephasing transformations of the lepton mixing matrices are

$$\begin{aligned} V &\to \begin{bmatrix} e^{-i\Phi_\nu} & 0 \\ 0 & U^{-1}_{n_g-n'_g} \end{bmatrix} V\ e^{i\Phi_E}, \\ W &\to e^{-i\Phi_\nu}\ W\ \eta_N, \end{aligned} \tag{3.13}$$

instead of eq. (3.4).

The additional unitary transformation matrix in eq. (3.13) can be used to eliminate parameters in V. The parameter counting for $n_g' < n_g$ is summarized in table 3. The number of CP-even parameters is $(n_g n_g' + n_g + n_g')$ and the number of CP-odd parameters is $(n_g n_g' - n_g)$, consistent with the results of ref. [21].

3.1.3 $n_g' > n_g$

For $n_g' > n_g$, the $n_g' \times n_g$ diagonal matrix Λ_ν can be written as

$$\Lambda_\nu \equiv \begin{bmatrix} \bar{\Lambda}_\nu \\ 0 \end{bmatrix}, \tag{3.14}$$

where 0 denotes the $(n_g' - n_g) \times n_g$ zero matrix, and $\bar{\Lambda}_\nu$ is a diagonal $n_g \times n_g$ matrix with n_g real positive eigenvalues. This matrix is invariant under

$$\begin{bmatrix} \bar{\Lambda}_\nu \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} e^{-i\Phi_\nu} & 0 \\ 0 & U_{n_g'-n_g} \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_\nu \\ 0 \end{bmatrix} e^{i\Phi_\nu}, \tag{3.15}$$

where $U_{n_g'-n_g}$ denotes an arbitrary $(n_g' - n_g) \times (n_g' - n_g)$ unitary matrix. The rephasing transformation of the lepton mixing matrices is

$$\begin{aligned} V &\rightarrow e^{-i\Phi_\nu}\, V\, e^{i\Phi_E}, \\ W &\rightarrow \begin{bmatrix} e^{-i\Phi_\nu} & 0 \\ 0 & U_{n_g'-n_g} \end{bmatrix} W\, \eta_N, \end{aligned} \tag{3.16}$$

instead of eq. (3.4).

The additional unitary transformation matrix in eq. (3.16) can be used to eliminate parameters in W. The parameter counting for $n_g' > n_g$ is summarized in table 4. The number of CP-even parameters is $(n_g n_g' + n_g + n_g')$ and the number of CP-odd parameters is $(n_g n_g' - n_g)$, consistent with the results of ref. [21].

3.2 Low-energy effective theory

The flavor matrices in the low-energy effective theory are written in eq. (2.10) in the weak eigenstate basis. These matrices are related to the mass eigenstate basis by

$$\begin{aligned} Y_U &= \mathsf{U}_{U^c}\, \Lambda_U\, \mathsf{U}_U, \\ Y_D &= \mathsf{U}_{D^c}\, \Lambda_D\, \mathsf{U}_D, \\ Y_E &= \mathsf{U}_{E^c}\, \Lambda_E\, \mathsf{U}_E, \\ C_5 &= \mathsf{U}_\nu'^T\, \Lambda_5\, \mathsf{U}_\nu'. \end{aligned} \tag{3.17}$$

Performing chiral flavor transformations in the low-energy theory with ${\mathcal{U}_{U^c}}^T = {\mathsf{U}_{U^c}}^{-1}$, ${\mathcal{U}_{D^c}}^T = {\mathsf{U}_{D^c}}^{-1}$, ${\mathcal{U}_{E^c}}^T = {\mathsf{U}_{E^c}}^{-1}$, $\mathcal{U}_Q = {\mathsf{U}_U}^{-1}$, $\mathcal{U}_L = {\mathsf{U}_E}^{-1}$ brings the flavor matrices to the form

$$
\begin{aligned}
Y_U &= \Lambda_U, \\
Y_D &= \Lambda_D\ V_{\rm CKM}^{-1}, \\
Y_E &= \Lambda_E, \\
C_5 &= \left(U_{\rm PMNS}^{-1}\right)^T\ \Lambda_5\ U_{\rm PMNS}^{-1},
\end{aligned}
\tag{3.18}
$$

where $V_{\rm CKM} \equiv \mathsf{U}_U {\mathsf{U}_D}^{-1}$ and $\mathsf{U}_{\rm PMNS}^{-1} \equiv \mathsf{U}'_\nu {\mathsf{U}_E}^{-1}$ are the two unitary matrices which describe flavor mixing in the low-energy effective theory. $V_{\rm CKM}$ is the CKM mixing matrix in the quark sector. $U_{\rm PMNS}$ is the PMNS mixing matrix in the lepton sector, which is the lepton mixing matrix which is physically measurable at low energies.

The real diagonal matrices $\Lambda_{U,D,E,5}$ are invariant under the rephasings

$$
\begin{aligned}
\Lambda_\psi &\to e^{-i\Phi_\psi}\ \Lambda_\psi\ e^{i\Phi_\psi},\ \ \psi = U, D, E, \\
\Lambda_5 &\to \eta_\nu\ \Lambda_5\ \eta_\nu,
\end{aligned}
\tag{3.19}
$$

which correspond to arbitrary phase redefinitions of the fermion mass eigenstate fields U^c, D^c, E^c, U, D and E, and the discrete rephasings $\nu \to \eta_\nu \nu$, where η_ν is a diagonal matrix with allowed eigenvalues ± 1 for the low-energy Majorana neutrino fields. Under these rephasings, the mixing matrices of the effective theory transform as

$$
\begin{aligned}
V_{\rm CKM} &\to e^{-i\Phi_U}\ V_{\rm CKM}\ e^{i\Phi_D}, \\
U_{\rm PMNS} &\to e^{-i\Phi_E}\ U_{\rm PMNS}\ \eta_\nu\,.
\end{aligned}
\tag{3.20}
$$

The quark mixing matrix $V_{\rm CKM}$ has the angles and phases given in table 1 as before. The counting of parameters in the lepton sector is summarized in table 5, and is well-known. $U_{\rm PMNS}$ contains $n_g(n_g-1)/2$ angles θ_i. The number of phases of $U_{\rm PMNS}$ is $n_g(n_g+1)/2$ minus the n_g phase redefinitions Φ_E, for a total of $n_g(n_g-1)/2$ phases consisting of $(n_g-1)(n_g-2)/2$ phases δ_i and (n_g-1) phases ψ_i. The canonical parametrization of $U_{\rm PMNS}$ is

$$
U_{\rm PMNS} = \mathcal{V}(\theta_i, \delta_i)\ e^{i\Psi/2}\,,
\tag{3.21}
$$

$\Psi = {\rm diag}(0, \psi_2, \ldots, \psi_n)$.

For $n_g = 3$, the low-energy lepton mixing matrix is given by

$$
U_{\rm PMNS} = \mathcal{V}\left(\theta_1^{(U)}, \theta_2^{(U)}, \theta_3^{(U)}, \delta^{(U)}\right) \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\psi_2^{(U)}/2} & 0 \\ 0 & 0 & e^{i\psi_3^{(U)}/2} \end{pmatrix},
\tag{3.22}
$$

where the superscript (U) denotes quantities in the PMNS matrix.

4 Invariant theory

In the previous sections, we have discussed the parameters (masses, angles and phases) for the low- and high-energy theories. We would like to analyze the theories using invariant

Matrices	Masses	Angles	Phases
Λ_E	n_g	0	0
Λ_5	n_g	0	0
U_{PMNS}	0	$\frac{1}{2}n_g(n_g-1)$	$\frac{1}{2}n_g(n_g-1)$
Total	$2n_g$	$\frac{1}{2}n_g(n_g-1)$	$\frac{1}{2}n_g(n_g-1)$

Table 5. Parameters in the lepton sector of the low-energy effective theory for n_g generations. The Λ_E and Λ_5 rows give the parameters if m_E or m_5 are considered separately. The U_{PMNS} row gives the mixing angles and phases of the PMNS mixing matrix.

quantities written directly in terms of the original parameters of the theory, the matrices $Y_{U,D,E,\nu}$ and M. The structure of the invariants is highly non-trivial, and depends in an interesting way on the number of generations.

To study the invariants, it is useful to introduce several mathematical results from invariant theory [16–18]. The general problem is the following: one has a set of variables $x_1, \ldots x_n$ which transform (reducibly or irreducibly) under the action of a group G. The set of polynomials in $\{x_i\}$ with complex coefficients form a ring $\mathbb{C}[x_1, \ldots, x_n]$. The polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is a free ring on the generators $x_1, \ldots, x_n$, i.e. it is given by taking linear combinations of all possible products of powers of the generators with coefficients in $\mathbb{C}$, and there are no non-trivial relations among the generators.

The ring $\mathbb{C}[x_1, \ldots, x_n]^G \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is the set of G-invariant polynomials, i.e. those polynomials which are unchanged by the action of G. This is clearly a ring, since sums and products of invariant polynomials are also invariant polynomials. A highly non-trivial result, if G is a reductive group,[5] is that $\mathbb{C}[x_1, \ldots, x_n]^G$ is finite generated. Let the generators be $I_1, \ldots, I_r$, each of which is a G-invariant polynomial in the original variables $x_1, \ldots, x_n$. Then, any G-invariant polynomial can be written as a polynomial $P \in \mathbb{C}[I_1, \ldots, I_r]$. However, $\mathbb{C}[x_1, \ldots, x_n]^G$ need not be a free ring in the generators $I_1, \ldots, I_r$; there can be non-trivial relations among them.

In the following sections, we analyze the invariant ring for the quark and lepton sectors of the Standard Model effective theory and the seesaw model. It is useful to first look at some simple examples before discussing the case of interest. We start with a famous result on symmetric polynomials, and then discuss three examples involving continuous groups which are closer in structure to the quark and lepton invariant problem. The first model is a theory which has a freely generated ring, with no relations. The second theory has one non-trivial relation, and is similar in structure to the ring for quark invariants for three generations studied in section 5.2 and for lepton invariants in the Standard Model for two generations studied in section 6.1. The third example is only slightly more complicated, but leads to an intricate structure of invariants, with many relations, and a complicated

[5] A reductive group is defined by the property that every representation is completely reducible. A Lie group which is a direct product of simple compact Lie groups and U(1) factors is reductive, as is any finite group.

Hilbert series. This is similar to what we find for lepton invariants in the Standard Model for three generations, and in the seesaw model for two and three generations.

4.1 Symmetric polynomials

The classic example from invariant theory is the study of symmetric polynomials. The permutation group S_n acts on a polynomial $f(x_1,\ldots,x_n)$ in $\mathbb{C}[x_1,\ldots,x_n]$ by

$$p : f(x_1,\ldots,x_n) \to f(x_{p(1)},\ldots,x_{p(n)}) \tag{4.1}$$

where $(p(1),\ldots,p(n))$ is a permutation of $(1,\ldots,n)$. A polynomial in $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$ is invariant under the action of any permutation. A standard result [23] is that the invariant ring is generated by the elementary symmetric polynomials

$$\begin{aligned}
I_1 &= x_1 + x_2 + \cdots x_n = \sum_i x_i, \\
I_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n = \sum_{i<j} x_ix_j, \\
I_3 &= x_1x_2x_3 + \cdots + x_{n-2}x_{n-1}x_n = \sum_{i<j<k} x_ix_jx_k, \\
&\vdots \\
I_n &= x_1x_2\ldots x_n.
\end{aligned} \tag{4.2}$$

In other words, any symmetric polynomial $f(x_1,\ldots,x_n)$ can be written as a *polynomial* in $I_1,\ldots,I_n$, $f(x_1,\ldots,x_n) = g(I_1,\ldots,I_n)$, e.g.

$$x_1^2 + x_2^2 + \cdots x_n^2 = I_1^2 - 2I_2\,. \tag{4.3}$$

The important point is that $g(I_1,\ldots,I_n)$ is a polynomial — otherwise the result would be trivial, for knowing $I_1,\ldots,I_n$, one could solve eq. (4.2) to find $x_1,\ldots,x_n$, and hence determine f.

4.2 Model I

Consider a theory with two couplings m_1 and m_2 which transform under a $G = \mathrm{U}(1)\times\mathrm{U}(1)$ symmetry as

$$m_1 \to e^{i\phi_1} m_1, \qquad m_2 \to e^{i\phi_2} m_2\ . \tag{4.4}$$

We look at the ring $\mathbb{C}[m_1,m_1^*,m_2,m_2^*]^{\mathrm{U}(1)\times\mathrm{U}(1)}$ of all polynomials which are $\mathrm{U}(1)\times\mathrm{U}(1)$ invariant. It is clear that they can be written as linear combinations of monomials of the form

$$(m_1m_1^*)^{r_1}\,(m_2m_2^*)^{r_2} \tag{4.5}$$

where r_1 and r_2 are integers. Thus, the ring of invariant polynomials is generated by the invariants $I_1 = m_1m_1^*$ and $I_2 = m_2m_2^*$, and there are no relations between these generators.

The Hilbert series $H(q)$ is defined as

$$H(q) = \sum_{r=0}^{\infty} c_r q^r \tag{4.6}$$

where c_r is the number of invariants of degree r, and $c_0 = 1$. In our example, $c_1 = 0$; $c_2 = 2$ since $m_1 m_1^*$ and $m_2 m_2^*$ are the two degree-two invariants; $c_3 = 0$; $c_4 = 3$ since $(m_1 m_1^*)^2$, $(m_1 m_1^*)(m_2 m_2^*)$ and $(m_2 m_2^*)^2$ are the three degree-four invariants; and so on. It is easy to see that the Hilbert series is

$$\begin{aligned} H(q) &= 1 + 2q^2 + 3q^4 + 4q^6 + 5q^8 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1) q^{2n} \\ &= \frac{1}{(1-q^2)^2} \,. \end{aligned} \tag{4.7}$$

Another derivation of the Hilbert series is the following. The generators $I_1 = m_1 m_1^*$ and $I_2 = m_2 m_2^*$ are both of degree two, and the invariants of higher order are given by multiplying together arbitrary powers of I_1 and I_2. The product

$$\left(1 + I_1 + I_1^2 + \cdots\right)\left(1 + I_2 + I_2^2 + \cdots\right) \tag{4.8}$$

gives each invariant once, which leads to the Hilbert series

$$H(q) = \left(1 + q^2 + q^4 + \cdots\right)\left(1 + q^2 + q^4 + \cdots\right) = \frac{1}{(1-q^2)^2}, \tag{4.9}$$

in agreement with eq. (4.7).

In the general case of a semisimple Lie group, it is known that $H(q)$ has the rational form

$$H(q) = \frac{N(q)}{D(q)}, \tag{4.10}$$

where the numerator $N(q)$ and denominator $D(q)$ are polynomials. Furthermore, the numerator is of degree d_N and is of the form

$$N(q) = 1 + c_1 q + \cdots c_{d_N - 1} q^{d_N - 1} + q^{d_N} \tag{4.11}$$

where the coefficients are non-negative, $c_r \geq 0$, and $N(q)$ is palindromic, i.e.

$$N(q) = q^{d_N} N(1/q), \tag{4.12}$$

or, more simply stated,

$$c_r = c_{d_N - r}. \tag{4.13}$$

The denominator takes the form

$$D(q) = \prod_{r=1}^{p} (1 - q^{d_r}), \tag{4.14}$$

and is of degree $d_D = \sum_r d_r$. The number of denominator factors p is equal to the number of parameters. The number of parameters is defined as the minimal codimension of an orbit, and agrees with the usual physics usage of the term.

Model I has $p = 2$ parameters, because we start with four objects m_1, m_2, m_1^* and m_2^* (or equivalently, the real and imaginary parts of m_1 and m_2), and have two phase redefinitions eq. (4.4), which eliminates two variables. In other words, one can always make a phase redefinition to make m_1 and m_2 real and non-negative, and these are the two independent parameters. In our example, $N(q) = 1$, $d_1 = d_2 = 2$ and the number of denominator factors is two. The number of denominator factors p is equal to the number of parameters.

There is a theorem due to Knop [24] which says that

$$\dim V \geq d_D - d_N \geq p \tag{4.15}$$

where $\dim V$ is the dimension of the vector space on which the group transformations act; d_D and d_N are the degrees of the denominator and numerator; and p is the number of parameters. In most cases, the upper bound is an equality, but not always. (We will see an example for the quark invariants involving only the U-quark mass matrix.) In Model I, the vector space basis is m_1, m_1^*, m_2, m_2^*, so $\dim V = 4$, $p = 2$, $d_N = 0$ and $d_D = \sum d_r = 4$, and we see that Knop's theorem gives $4 \geq 4 - 0 \geq 2$, with an equality for the upper bound.

One also can construct a multi-graded Hilbert series. Let $c_{r_1 r_2 r_3 r_4}$ be the number of invariants of order r_1 in m_1, order r_2 in m_1^*, order r_3 in m_2, and order r_4 in m_2^*. Then

$$h(q_1, q_2, q_3, q_4) = \sum c_{r_1 r_2 r_3 r_4} q_1^{r_1} q_2^{r_2} q_3^{r_3} q_4^{r_4} = \frac{1}{(1 - q_1 q_2)(1 - q_3 q_4)}, \tag{4.16}$$

and the usual Hilbert series is $H(q) = h(q, q, q, q)$. The multi-graded series gives more information about the structure of the invariants. However, it is important to remember that the results quoted above for $H(q)$, eqs. (4.10)–(4.15), do not hold in general for the multi-graded case.

The Hilbert series provides far more information than the number of invariants of each degree, given by the series expansion eq. (4.6). It encodes the structure of the invariant ring and the form of the relations between invariants, as will be seen from the examples considered below. Furthermore, the properties of the Hilbert series, such as eqs. (4.13), (4.14), (4.15) provide a strong constraint on the number of invariants. Computing invariants to high orders is, in general, a difficult task (i.e. the problem is not of polynomial complexity). One can determine the Hilbert series, and hence the invariants to arbitrarily high order, by computing some of the expansion coefficients c_r, and using the constraints on the Hilbert series to determine $H(q)$. There is no systematic procedure for doing this. One computes the number of invariants c_r of degree r for some values of r, which provide clues to the form of the numerator and denominator of the Hilbert series. Eventually, only a unique possible answer remains.

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4.3 Model II

Consider a theory with couplings m_1 and m_2 with charges one and two, respectively, under a $G = \mathrm{U}(1)$ symmetry,

$$m_1 \to e^{i\phi} m_1, \qquad m_2 \to e^{2i\phi} m_2 \ . \tag{4.17}$$

The ring of invariant polynomials $\mathbb{C}[m_1, m_1^*, m_2, m_2^*]^{\mathrm{U}(1)}$ is generated by the four basic invariants $I_1 = m_1 m_1^*$, $I_2 = m_2 m_2^*$, $I_3 = m_2 m_1^{*2}$ and $I_4 = m_2^* m_1^2$. These generators, however, are not all independent, since $I_3 I_4 = I_1^2 I_2$, so that $\mathbb{C}[m_1, m_1^*, m_2, m_2^*]^{\mathrm{U}(1)}$ is not a free ring generated by I_1 through I_4.

It is straightforward to show that the multi-graded Hilbert series is

$$h(q_1, q_2, q_3, q_4) = \frac{1 - q_1^2 q_2^2 q_3 q_4}{(1 - q_1 q_2)(1 - q_3 q_4)(1 - q_3 q_2^2)(1 - q_4 q_1^2)}, \tag{4.18}$$

where q_1, q_2, q_3 and q_4 count powers of m_1, m_1^*, m_2 and m_2^*, respectively.

The denominator of the multi-graded Hilbert series is generated by the invariants I_1 through I_4, whereas the numerator compensates for the fact that $I_3 I_4 = I_1^2 I_2$ counts as only one invariant at order $q_1^2 q_2^2 q_3 q_4$. The numerator of the multi-graded Hilbert series does not have the special properties of the numerator of the Hilbert series $H(q)$ discussed in the previous example.

In this example, $\dim V = 4$, $\dim G = 1$, and there are three parameters since the phase transformation eq. (4.17) eliminates one of the original four real variables in m_1 and m_2. The Hilbert series $H(q) = h(q, q, q, q)$ is

$$H(q) = \frac{1 + q^3}{(1 - q^2)^2 (1 - q^3)}, \tag{4.19}$$

which has a palindromic numerator with $d_N = 3$, and a denominator with $d_D = 7$, and $p = 3$ is equal to the number of denominator factors and to the number of parameters. Knop's theorem gives $4 \geq 7 - 3 \geq 3$, with an equality for the upper bound.

Expanding eq. (4.19) in a series in q gives the number of invariants of each degree. We see that there are two generators of degree two, I_1 and I_2, and one generator of degree three, which can be chosen to be $I_3 + I_4$, corresponding to the denominator factors $(1 - q^2)^2$ and $(1 - q^3)$, respectively. Expanding out the denominator would give a coefficient of q^3 of $+1$. There are two invariants of degree three, $I_3 \pm I_4$. The missing degree-three invariant $I_3 - I_4$ is counted by the $+q^3$ term in the numerator, so that the coefficient of q^3 in the expansion of $H(q)$ is 2. When the denominator factors are expanded in a series, they can occur to any power, so one can have arbitrary powers of I_1, I_2 and $I_3 + I_4$. However, the q^3 factor in the numerator occurs only once. This means that powers of $I_3 - I_4$ higher than the first can all be eliminated in terms of polynomials $P(I_1, I_2, I_3 + I_4)$ which have already been included. This statement follows from the identity

$$(I_3 - I_4)^2 = (I_3 + I_4)^2 - 4 I_3 I_4 = (I_3 + I_4)^2 - 4 I_1^2 I_2. \tag{4.20}$$

There exists a similar identity for the Jarlskog invariant which will be derived in section 5.

The generator $I_3 + I_4$ of the denominator is not homogeneous in the multi-grading; I_3 is of degree $q_3 q_2^2$ and I_4 is of degree $q_4 q_1^2$, which is why eq. (4.18) can not be written in a form similar to eq. (4.19) with positive coefficients in the numerator and one less generator in the denominator.

4.4 Model III

Consider yet another model with three couplings m_1, m_2 and m_3 with charges 1, 2 and 3, respectively, under a U(1) symmetry,

$$m_1 \to e^{i\phi} m_1,\ m_2 \to e^{2i\phi} m_2,\ m_3 \to e^{3i\phi} m_3\ . \tag{4.21}$$

The structure of the invariants is considerably more complicated than in the previous examples, even though the theory is only slightly more complicated. All the invariant polynomials are generated by thirteen invariant generators

$$\begin{aligned}
I_1 &= m_1 m_1^*,\\
I_2 &= m_2 m_2^*,\\
I_3 &= m_3 m_3^*,\\
I_4 &= m_1^2 m_2^*,\\
I_5 &= m_1^{*2} m_2,\\
I_6 &= m_1^3 m_3^*,\\
I_7 &= m_1^{*3} m_3,\\
I_8 &= m_2^3 m_3^{*2},\\
I_9 &= m_2^{*3} m_3^2,\\
I_{10} &= m_1 m_2 m_3^*,\\
I_{11} &= m_1^* m_2^* m_3,\\
I_{12} &= m_1 m_3 m_2^{*2},\\
I_{13} &= m_1^* m_3^* m_2^2.
\end{aligned} \tag{4.22}$$

There are 35 relations between products of invariants $I_i I_j$ given by: $I_4 I_5 = I_1^2 I_2$, $I_4 I_7 = I_1^2 I_{11}$, $I_4 I_8 = I_2 I_{10}^2$, $I_4 I_9 = I_{12}^2$, $I_4 I_{10} = I_2 I_6$, $I_4 I_{11} = I_1 I_{12}$, $I_4 I_{13} = I_1 I_2 I_{10}$, $I_5 I_6 = I_1^2 I_{10}$, $I_5 I_8 = I_{13}^2$, $I_5 I_9 = I_2 I_{11}^2$, $I_5 I_{10} = I_1 I_{13}$, $I_5 I_{11} = I_2 I_7$, $I_5 I_{12} = I_1 I_2 I_{11}$, $I_6 I_7 = I_1^3 I_3$, $I_6 I_8 = I_{10}^3$, $I_6 I_9 = I_3 I_4 I_{12}$, $I_6 I_{11} = I_1 I_3 I_4$, $I_6 I_{12} = I_3 I_4^2$, $I_6 I_{13} = I_1 I_{10}^2$, $I_7 I_8 = I_3 I_5 I_{13}$, $I_7 I_9 = I_{11}^3$, $I_7 I_{10} = I_1 I_3 I_5$, $I_7 I_{12} = I_1 I_{11}^2$, $I_7 I_{13} = I_3 I_5^2$, $I_8 I_9 = I_2^3 I_3^2$, $I_8 I_{11} = I_2 I_3 I_{13}$, $I_8 I_{12} = I_2^2 I_3 I_{10}$, $I_9 I_{10} = I_2 I_3 I_{12}$, $I_9 I_{13} = I_2^2 I_3 I_{11}$, $I_{10} I_{11} = I_1 I_2 I_3$, $I_{10} I_{12} = I_2 I_3 I_4$, $I_{10} I_{13} = I_1 I_8$, $I_{11} I_{12} = I_2 I_3 I_5$, $I_{11} I_{13} = I_2 I_3 I_5$ and $I_{12} I_{13} = I_1 I_2^2 I_3$. The new feature here is that these relations are not independent — there are relations among the relations (known as syzygies in the mathematics literature), e.g. multiplying both sides of $I_4 I_7 = I_1^2 I_{11}$ and $I_5 I_6 = I_1^2 I_{10}$ gives

$$I_4 I_5 I_6 I_7 = I_1^4 I_{10} I_{11}\,, \tag{4.23}$$

which is also obtained by multiplying the relations $I_4 I_5 = I_1^2 I_2$ and $I_6 I_7 = I_1^3 I_3$, and using $I_{10} I_{11} = I_1 I_2 I_3$. The Hilbert series is

$$H(q) = \frac{1 + q^2 + 3q^3 + 4q^4 + 4q^5 + 4q^6 + 3q^7 + q^8 + q^{10}}{(1-q^2)^2(1-q^3)(1-q^4)(1-q^5)} \,. \tag{4.24}$$

Here $\dim V = 6$, $\dim G = 1$, and the number of parameters is 5. From the Hilbert series, $d_N = 10$, $d_D = 16$, and $p = 5$. Knop's theorem gives $6 \geq 16 - 10 \geq 5$, with an equality for the upper bound.

There are thirteen invariants in eq. (4.22). However, there are only five denominator factors in eq. (4.24), so only five basic invariants, two of degree two, and one each of degrees three, four and five, generate a free ring. The other invariants must satisfy non-trivial relations (those given below eq. (4.22)), and this is reflected by the complicated numerator in eq. (4.24), which implies that the invariant ring has a non-trivial structure, with many relations. The different terms in the numerator show that there are many invariants which can be eliminated when raised to higher powers, or multiplied by lower order invariants, by relations analogous to eq. (4.20). There is one invariant of degree two (the $+q^2$ term), three in degree three (the $+3q^3$ term), etc. This model shows that even a relatively simple theory can lead to a set of invariants with a complicated syzygy structure. The number of invariants and relations of each degree is encoded in the Hilbert series.

5 Quark invariants

We can now address the first problem of interest — flavor invariants in the quark sector. We are interested in polynomials in m_U, ${m_U}^\dagger$, m_D and ${m_D}^\dagger$ where

$$\begin{aligned} m_U &\to {\mathcal{U}_{U^c}}^T \, m_U \, \mathcal{U}_Q, \\ m_D &\to {\mathcal{U}_{D^c}}^T \, m_D \, \mathcal{U}_Q, \end{aligned} \tag{5.1}$$

under the chiral flavor transformations.[6] To cancel $\mathcal{U}_{U^c}$ and $\mathcal{U}_{D^c}$, one must consider the combinations

$$\begin{aligned} X_U &\equiv {m_U}^\dagger m_U, \\ X_D &\equiv {m_D}^\dagger m_D, \end{aligned} \tag{5.2}$$

which both transform as adjoints

$$X_{U,D} \to \mathcal{U}_Q^\dagger \, X_{U,D} \, \mathcal{U}_Q. \tag{5.3}$$

Thus, the invariants are traces of products of X_U and X_D. The structure of the invariants depends non-trivially on the number of generations, so we consider the cases $n_g = 2$ and $n_g = 3$ separately.

[6] One could equally well work with the Yukawa matrices, which differ by factor $v/\sqrt{2}$.

5.1 $n_g = 2$

First, consider invariants involving only X_U. The basic invariants are

$$\langle X_U \rangle \,, \langle {X_U}^2 \rangle \,, \langle {X_U}^3 \rangle \,, \ldots \tag{5.4}$$

where $\langle * \rangle$ denotes a matrix trace. This series of traces terminates after n_g terms for an $n_g \times n_g$ matrix, by the Cayley-Hamilton theorem which states that every matrix satisfies its characteristic equation. For an arbitrary 2×2 matrix A, the Cayley-Hamilton theorem gives

$$A^2 = \langle A \rangle \, A + \tfrac{1}{2} \, \left[\langle A^2 \rangle - \langle A \rangle^2 \right] \mathbb{1}. \tag{5.5}$$

Taking the trace of both sides gives the trivial result $\langle A^2 \rangle = \langle A^2 \rangle$. Multiplying by A and taking the trace implies

$$\langle A^3 \rangle = \tfrac{3}{2} \langle A \rangle \langle A^2 \rangle - \tfrac{1}{2} \langle A \rangle^3 \,, \tag{5.6}$$

so that $\langle A^n \rangle$, $n \geq 3$ can be written in terms of $\langle A \rangle$ and $\langle A^2 \rangle$. Thus, there are two independent invariants, $I_{2,0} = \langle X_U \rangle$ and $I_{4,0} = \langle {X_U}^2 \rangle$, which can be constructed from X_U alone. Both of these invariants are CP even. The two invariants contain the same information as the eigenvalues of X_U, i.e. the two U-type quark masses. For invariants constructed only from m_U, the number of parameters is $p = 2$, the two eigenvalues of X_U. The vector space has $\dim V = 8$, because m_U and ${m_U}^\dagger$ are both 2×2 matrices, and $I_{2,0}$ and $I_{4,0}$ are of degree two and four, respectively, in m_U, so the Hilbert series is

$$H(q) = \frac{1}{(1-q^2)(1-q^4)}. \tag{5.7}$$

Here $d_N = 0$, $d_D = 6$ are the degrees of the numerator and denominator, respectively, and the number of denominator factors is $p = 2$, which is equal to the number of parameters. Knop's theorem gives $8 \geq 6 - 0 \geq 2$, which holds, but this time the upper bound is not an equality.

Similarly, there are two independent CP-even invariants $I_{0,2} = \langle X_D \rangle$ and $I_{0,4} = \langle {X_D}^2 \rangle$ which involve only X_D. These two invariants contain the same information as the eigenvalues of X_D, namely the two D-type quark masses.

Invariants containing both X_U and X_D can be written as traces of the form

$$\langle {X_U}^{r_1} {X_D}^{s_1} {X_U}^{r_2} {X_D}^{s_2} \ldots \rangle \,, \tag{5.8}$$

for integers r_i and s_i. The Cayley-Hamilton theorem for a 2×2 matrix, eq. (5.5), implies that all powers r_i and s_i greater than one in eq. (5.8) can be reduced, so we are left with traces of the form

$$\langle X_U X_D \ldots X_U X_D \rangle = \langle (X_U X_D)^r \rangle \,. \tag{5.9}$$

Again, invariants with $r > 1$ can be rewritten in terms of lower order invariants, so there is only one independent invariant, $I_{2,2} = \langle X_U X_D \rangle$, which is CP even.

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In summary, the basic quark invariants for $n_g = 2$ quark generations, which generate all the invariants, are:

$$\begin{aligned}
I_{2,0} &= \langle X_U \rangle = \langle {m_U}^\dagger m_U \rangle \,, \\
I_{0,2} &= \langle X_D \rangle = \langle {m_D}^\dagger m_D \rangle \,, \\
I_{4,0} &= \langle {X_U}^2 \rangle = \langle \left({m_U}^\dagger m_U \right)^2 \rangle \,, \\
I_{2,2} &= \langle X_U X_D \rangle = \langle {m_U}^\dagger m_U {m_D}^\dagger m_D \rangle \,, \\
I_{0,4} &= \langle {X_D}^2 \rangle = \langle \left({m_D}^\dagger m_D \right)^2 \rangle \,.
\end{aligned} \tag{5.10}$$

Writing the invariants in terms of the usual quark masses and the Cabibbo angle gives

$$\begin{aligned}
I_{2,0} &= m_u^2 + m_c^2, \\
I_{0,2} &= m_d^2 + m_s^2, \\
I_{4,0} &= m_u^4 + m_c^4, \\
I_{2,2} &= m_u^2 m_s^2 + m_c^2 m_d^2 + (m_s^2 - m_d^2)(m_c^2 - m_u^2)\cos^2\theta, \\
I_{0,4} &= m_d^4 + m_s^4.
\end{aligned} \tag{5.11}$$

Knowing the five invariants allows one to determine the four masses and θ, because $m_i \ge 0$, and θ lies in the first quadrant.

Using u and d to count powers of m_U and m_D gives the multi-graded Hilbert series

$$h(u,d) = \frac{1}{(1-u^2)(1-u^4)(1-d^2)(1-d^4)(1-u^2d^2)}. \tag{5.12}$$

The Hilbert series $H(q) = h(q,q)$ is

$$H(q) = \frac{1}{(1-q^2)^2(1-q^4)^3}. \tag{5.13}$$

In this example, $p = 5$ (four masses and one mixing angle, see table 1), $\dim V = 16$, since there are four 2×2 matrices, $d_N = 0$, and $d_D = 16$. The number of denominator factors is the number of parameters, and Knop's theorem gives $16 \ge 16 - 0 \ge 5$, with the upper bound an equality.

The denominator factors in eq. (5.13) show that there are two generators of degree two, and three of degree four, which agrees with eq. (5.10).

If one started with X_U and X_D as the basic objects, then $\dim V = 8$. In this case, the Hilbert series is given by replacing $q^2 \to q$ in eq. (5.13), since we now count powers of X_U, X_D rather than m_U, m_D, so $d_N = 0$, $d_D = 8$ and Knop's inequality becomes $8 \ge 8 - 0 \ge 5$.

5.2 $n_g = 3$

For an arbitrary 3×3 matrix A, the Cayley-Hamilton theorem states that

$$A^3 = A^2 \langle A \rangle - \frac{1}{2} A \left[\langle A \rangle^2 - \langle A^2 \rangle \right] + \frac{1}{6} \left[\langle A \rangle^3 - 3 \langle A^2 \rangle \langle A \rangle + 2 \langle A^3 \rangle \right] \mathbb{1}. \tag{5.14}$$

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Taking the trace of both sides gives the trivial result $\langle A^3\rangle = \langle A^3\rangle$. Multiplying by A and taking the trace gives

$$\langle A^4\rangle = \frac{1}{6}\langle A\rangle^4 - \langle A\rangle^2\langle A^2\rangle + \frac{4}{3}\langle A^3\rangle\langle A\rangle + \frac{1}{2}\langle A^2\rangle^2, \tag{5.15}$$

so that $\langle A^n\rangle$, $n \geq 4$ can be rewritten in terms of $\langle A\rangle$, $\langle A^2\rangle$, and $\langle A^3\rangle$.

Thus, the invariants involving X_U alone are $I_{2,0} = \langle X_U\rangle$, $I_{4,0} = \langle {X_U}^2\rangle$ and $I_{6,0} = \langle {X_U}^3\rangle$, and invariants involving X_D alone are $I_{0,2} = \langle X_D\rangle$, $I_{0,4} = \langle {X_D}^2\rangle$ and $I_{0,6} = \langle {X_D}^3\rangle$, all of which are CP even.

Invariants containing both X_U and X_D are of the form eq. (5.8), but now with $r_i = 1, 2$ and $s_i = 1, 2$, so that one has traces of products of X_U, X_U^2, X_D, X_D^2. This restriction still leads to an infinite number of invariants. However, many of these invariants are not independent. For arbitrary 3×3 matrices A, B and C, one has the identity

$$\begin{aligned}
0 = &\langle A\rangle^2\langle B\rangle\langle C\rangle - \langle BC\rangle\langle A\rangle^2 - 2\langle AB\rangle\langle A\rangle\langle C\rangle \\
&-2\langle AC\rangle\langle A\rangle\langle B\rangle + 2\langle ABC\rangle\langle A\rangle + 2\langle ACB\rangle\langle A\rangle \\
&-\langle A^2\rangle\langle B\rangle\langle C\rangle + 2\langle AB\rangle\langle AC\rangle + \langle A^2\rangle\langle BC\rangle \\
&+2\langle C\rangle\langle A^2B\rangle + 2\langle B\rangle\langle A^2C\rangle - 2\langle A^2BC\rangle \\
&-2\langle A^2CB\rangle - 2\langle ABAC\rangle
\end{aligned} \tag{5.16}$$

which can be derived by substituting $A \to A + B + C$ into eq. (5.15), and picking out the order A^2BC terms. This identity eliminates $\langle ABAC\rangle$, i.e. traces where the same matrix is repeated, so that in invariants eq. (5.8), X_U, ${X_U}^2$, X_D and ${X_D}^2$ can each occur at most once. For example, $\langle X_U \ldots X_U \ldots\rangle$ can be replaced by $\langle X_U^2 \ldots\rangle$, and $\langle X_U^2 \ldots X_U^2 \ldots\rangle$ can be replaced by $\langle X_U^4 \ldots\rangle$, which can then be eliminated using eq. (5.14).

Writing out all of the possibilities gives the basic quark invariants for $n_g = 3$ quark generations. There are 11 CP-even invariants, ten of which are

$$\begin{aligned}
I_{2,0} &= \langle X_U\rangle\,, \\
I_{0,2} &= \langle X_D\rangle\,, \\
I_{4,0} &= \langle {X_U}^2\rangle\,, \\
I_{2,2} &= \langle X_U X_D\rangle\,, \\
I_{0,4} &= \langle {X_D}^2\rangle\,, \\
I_{6,0} &= \langle {X_U}^3\rangle\,, \\
I_{4,2} &= \langle {X_U}^2 X_D\rangle\,, \\
I_{2,4} &= \langle X_U {X_D}^2\rangle\,, \\
I_{0,6} &= \langle {X_D}^3\rangle\,, \\
I_{4,4} &= \langle {X_U}^2 {X_D}^2\rangle\,,
\end{aligned} \tag{5.17}$$

and one CP-odd invariant

$$I_{6,6}^{(-)} = \langle {X_U}^2 {X_D}^2 X_U X_D\rangle - \langle {X_D}^2 {X_U}^2 X_D X_U\rangle\,. \tag{5.18}$$

The eleventh CP-even invariant is

$$I_{6,6}^{(+)} = \langle {X_U}^2 {X_D}^2 X_U X_D \rangle + \langle {X_D}^2 {X_U}^2 X_D X_U \rangle \ . \tag{5.19}$$

All the invariants in the quark sector can be written as polynomials in these basic invariants.

The multi-graded and one-variable Hilbert series are

$$\begin{aligned} h(u,d) &= \frac{1+u^6d^6}{(1-u^2)(1-u^4)(1-u^6)(1-d^2)(1-d^4)(1-d^6)(1-u^2d^2)(1-u^4d^2)(1-u^2d^4)(1-u^4d^4)}, \\ H(q) = h(q,q) &= \frac{1+q^{12}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}, \end{aligned} \tag{5.20}$$

respectively. This case has $p = 10$ parameters, consisting of 6 masses, three angles and one phase, which agrees with the number of denominator factors. The original variable space has $\dim V = 36$, from the two 3×3 mass matrices and their complex conjugates. The degrees of the numerator and denominator are $d_N = 12$ and $d_D = 48$, respectively, and Knop's inequality is $36 \geq 48 - 12 \geq 10$, which is satisfied, with the upper bound being an equality. If one started with X_U and X_D as the basic objects, then $\dim V = 18$, and the Hilbert series is given by replacing $q^2 \to q$ in eq. (5.20), so $d_N = 6$, $d_D = 24$, and Knop's inequality becomes $18 \geq 24 - 6 \geq 10$.

The denominator of eq. (5.20) shows that there are two invariants of degree two, three of degree four, four of degree six, and one of degree eight, which can occur multiplied in arbitrary combinations, with no relations among them. One can see that their degrees match the denominator factors in eq. (5.20). What about the remaining two invariants? The numerator factor of eq. (5.20) shows that there is one additional invariant of degree twelve other than those given by products of denominator factors. This is the CP-odd invariant eq. (5.18). The Hilbert series implies that the other degree-twelve invariant, eq. (5.19), cannot be an independent invariant. Indeed, it can be written as a polynomial in the other CP-even invariants,

$$\begin{aligned} 3I_{6,6}^{(+)} &= I_{2,0}^3 I_{0,2}^3 - I_{2,0} I_{4,0} I_{0,2}^3 - 3I_{2,2} I_{2,0}^2 I_{0,2}^2 + 3I_{4,2} I_{2,0} I_{0,2}^2 - I_{0,4} I_{2,0}^3 I_{0,2} + 3I_{2,4} I_{2,0}^2 I_{0,2} \\ &\quad -3I_{4,4} I_{2,0} I_{0,2} + I_{0,4} I_{6,0} I_{0,2} + 3I_{2,4} I_{4,2} + 3I_{2,2} I_{4,4} + I_{0,6} I_{2,0} I_{4,0} - I_{0,6} I_{6,0}, \end{aligned} \tag{5.21}$$

and so can be eliminated.

The Hilbert series numerator only has an entry q^{12}, but there is no q^{24} term. This means that $I_{6,6}^{(-)}$ is an independent invariant, but the square and all higher powers of $I_{6,6}^{(-)}$ are not. The square of the CP-odd invariant $I_{6,6}^{(-)}$ is CP-even, and can be written as a polynomial (with 241 terms out of a possible 305 terms) in the CP-even invariants of eq. (5.17). The most general polynomial invariant in the quark sector can be written as

$$P_1 + I_{6,6}^{(-)} \ P_2 \tag{5.22}$$

where P_1 and P_2 are polynomials in the CP-even invariants eq. (5.17).

This example illustrates how the structure of the invariants is encoded in the Hilbert series. For many purposes, the details of the relations, such as eq. (5.21), or the formula

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for $\left(I_{6,6}^{(-)}\right)^2$ are not important; all one needs to know is that $I_{6,6}^{(-)}$ occurs linearly, and $I_{6,6}^{(+)}$ can be eliminated.

The quark sector parameters are determined by the ten CP-even parameters $I_{2,0}$, $I_{4,0}$, $I_{6,0}$, $I_{0,2}$, $I_{0,4}$, $I_{0,6}$, $I_{2,2}$, $I_{2,4}$, $I_{4,2}$, $I_{4,4}$, and the single CP-odd parameter $I_{6,6}^{(-)}$. From the CP-even invariants, one can determine the U-type quark masses $m_{u,c,t}$ and D-type quark masses $m_{d,s,b}$, which are real and non-negative, and four combinations of the CKM parameters, $\cos\theta_{12}$, $\cos\theta_{13}$, $\cos\theta_{23}$ and $\cos\delta$, all of which are CP even. Since the CKM angles θ_{12}, θ_{13}, θ_{23} lie in the first quadrant, these angles are determined uniquely by their cosines. However, $\cos\delta$ does not determine the phase δ uniquely, because it cannot distinguish between δ and $-\delta$. Under CP, $\delta \leftrightarrow -\delta$. Thus, one Z_2 piece of information, the sign of δ, is missing. This sign is provided by the invariant $I_{6,6}^{(-)}$. The only information needed is the sign of $I_{6,6}^{(-)}$, which is why $\left(I_{6,6}^{(-)}\right)^2$ can be written in terms of the other CP-even invariants. This discussion corresponds to the well-known result that the unitarity triangle can be obtained by measuring the lengths of its sides, which are CP-conserving, rather than the angles, which are CP-violating. Knowing the sides determines the triangle up to a two-fold reflection ambiguity, which is fixed by the sign of $I_{6,6}^{(-)}$, or, equivalently, the sign of the Jarlskog invariant, so that the only additional information contained in the Jarlskog invariant is the sign. The relations between the invariants are similar to those obtained by studying rephasing invariants [5].

The invariant $I_{6,6}^{(-)}$ also can be written as

$$I_{6,6}^{(-)} = \frac{1}{3}\left\langle [X_U, X_D]^3 \right\rangle , \tag{5.23}$$

and is proportional to the Jarlskog invariant J [2],

$$I_{6,6}^{(-)} = 2iJ(m_c^2-m_u^2)(m_t^2-m_c^2)(m_t^2-m_u^2)(m_s^2-m_d^2)(m_b^2-m_s^2)(m_b^2-m_d^2), \tag{5.24}$$

where

$$J = \text{Im}\, (V_{\text{CKM}})_{11}\, (V_{\text{CKM}})_{12}^*\, (V_{\text{CKM}})_{22}\, (V_{\text{CKM}})_{21}^* \,. \tag{5.25}$$

$I_{6,6}^{(-)}$ vanishes if two U-type quarks or two D-type quarks are degenerate. It is well-known that quark CP violation vanishes for degenerate U-type or D-type quarks. $I_{6,6}^{(-)}$ is odd under the exchange of two U-type or two D-type masses, e.g under $m_u \leftrightarrow m_c$, whereas the invariants in eq. (5.17) are even under exchange, so $I_{6,6}^{(-)}$ cannot be written in terms of the other invariants. $\left(I_{6,6}^{(-)}\right)^2$ is even under exchange, and can be written in terms of the other invariants.

It is, of course, well-known that CP conservation in the quark sector requires $J=0$, or equivalently, $I_{6,6}^{(-)}=0$. What is new is the structure of the ring of all invariant polynomials, and the relation between the CP-conserving and CP-violating invariants.

The invariants which must vanish in the quark sector for CP conservation was determined for an arbitrary number of generations in ref. [25].

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6 Lepton invariants for two generations

The structure of the lepton invariants, like the quark invariants, depends on the number of generations, so we first consider the case of $n_g = 2$ generations in this section. The case of $n_g = 3$ generations is considered in section 7. We will outline the derivation of the results, but not give all the details.

6.1 The standard model effective theory

We now study the lepton invariants in the Standard Model low-energy effective theory with a neutrino Majorana mass term. The structure of the lepton invariants is considerably more complicated than the quark invariants. The lepton sector of the low-energy theory contains the flavor symmetry breaking matrices Y_E and C_5, so we are interested in polynomials in m_E, ${m_E}^\dagger$, m_5 and ${m_5}^* = {m_5}^\dagger$, since m_5 is a symmetric matrix. These matrices transform as

$$\begin{aligned} m_E &\to {\mathcal{U}_{E^c}}^T \; m_E \, \mathcal{U}_L \, , \\ {m_E}^\dagger &\to {\mathcal{U}_{E^c}}^\dagger \; {m_E}^\dagger \, {\mathcal{U}_L}^* \, , \\ m_5 &\to {\mathcal{U}_L}^T \; m_5 \, \mathcal{U}_L \, , \\ {m_5}^* &\to {\mathcal{U}_L}^\dagger \; {m_5}^* \, {\mathcal{U}_L}^* \, , \end{aligned} \tag{6.1}$$

under chiral flavor transformations. To cancel $\mathcal{U}_{E^c}$, one must consider the combinations

$$\begin{aligned} X_E &\equiv {m_E}^\dagger m_E , \\ X_E^* = {X_E}^T &\equiv {m_E}^T {m_E}^* , \end{aligned} \tag{6.2}$$

which transform as

$$\begin{aligned} X_E &\to \mathcal{U}_L^\dagger \; X_E \, \mathcal{U}_L , \\ {X_E}^T &\to {\mathcal{U}_L}^T \; {X_E}^T \, {\mathcal{U}_L}^* \; . \end{aligned} \tag{6.3}$$

It also is convenient to define

$$X_5 \equiv {m_5}^* m_5 , \tag{6.4}$$

which transforms as

$$X_5 \to {\mathcal{U}_L}^\dagger \; X_5 \, \mathcal{U}_L \, , \tag{6.5}$$

as well as $\left({m_5}^* \;\; ({X_E}^n)^T \;\; m_5\right)$, which transforms as

$$\left({m_5}^* \;\; ({X_E}^n)^T \;\; m_5\right) \to {\mathcal{U}_L}^\dagger \left({m_5}^* \;\; ({X_E}^n)^T \;\; m_5\right) \, \mathcal{U}_L . \tag{6.6}$$

The invariants involving only X_E are $I_{2,0} = \langle X_E \rangle$ and $I_{4,0} = \langle {X_E}^2 \rangle$, whereas the invariants involving only m_5 and ${m_5}^*$ are $I_{0,2} = \langle X_5 \rangle$ and $I_{0,4} = \langle {X_5}^2 \rangle$.

The invariants involving X_E, m_5 and ${m_5}^*$ are of the form

$$\langle {m_5}^* \;\; ({X_E}^{r_1})^T \;\; m_5 \; {X_E}^{s_1} \; \ldots {m_5}^* \;\; ({X_E}^{r_n})^T \;\; m_5 \; {X_E}^{s_n} \rangle \tag{6.7}$$

for integers r_i and s_i. The Cayley-Hamilton theorem implies that all powers r_i and s_i greater than one in eq. (6.7) can be rewritten in terms of lower order invariants. Thus, one needs to consider traces of matrix products containing the matrices X_E, X_5, and $\left(m_5{}^* \ X_E{}^T \ m_5\right)$ at most once.

In summary, the generators of the invariants are:

$$\begin{aligned}
I_{2,0} &= \langle X_E \rangle = \langle m_E{}^\dagger m_E \rangle \,, \\
I_{0,2} &= \langle X_5 \rangle = \langle m_5{}^* m_5 \rangle \,, \\
I_{4,0} &= \langle X_E{}^2 \rangle = \langle \left(m_E{}^\dagger m_E \right)^2 \rangle \,, \\
I_{2,2} &= \langle m_5{}^* \ X_E{}^T \ m_5 \rangle = \langle m_5 \ X_E \ m_5{}^* \rangle = \langle m_E{}^T \ m_E{}^* \ m_5 \ m_5{}^* \rangle = \langle m_E{}^\dagger \ m_E \ m_5{}^* \ m_5 \rangle \,, \\
I_{0,4} &= \langle X_5{}^2 \rangle = \langle (m_5{}^* m_5)^2 \rangle \,, \\
I_{4,2} &= \langle m_5{}^* \ X_E{}^T \ m_5 \ X_E \rangle = \langle m_5{}^* \ m_E{}^T m_E{}^* \ m_5 \ m_E{}^\dagger m_E \rangle \,, \\
I_{4,4}^{(-)} &= \langle m_5{}^* \ X_E{}^T \ m_5 \ X_E \ m_5{}^* \ m_5 \rangle - \langle m_5{}^* \ X_E{}^T \ m_5 \ m_5{}^* \ m_5 \ X_E \rangle \\
&= \langle m_5{}^* \ m_E{}^T m_E{}^* \ m_5 \ m_E{}^\dagger m_E \ m_5{}^* \ m_5 \rangle - \langle m_5{}^* \ m_E{}^T m_E{}^* \ m_5 \ m_5{}^* \ m_5 \ m_E{}^\dagger m_E \rangle \,,
\end{aligned} \tag{6.8}$$

where $I_{4,4}^{(-)}$ is CP odd, and the rest are CP even. The square of the CP-odd invariant, $\left(I_{4,4}^{(-)}\right)^2$, is not independent; it can be expressed in terms of polynomials in the other CP-even invariants. In addition, the CP-even invariant $I_{4,4}^{(+)}$, obtained by the substitution $- \to +$ in $I_{4,4}^{(-)}$, is not independent, and thus is not included in the above list.

There are six parameters: four masses, one angle and one phase, see table 5. The four masses, one mixing angle, and one phase, can be determined from $I_{2,0}$, $I_{4,0}$, $I_{0,2}$, $I_{0,4}$, $I_{2,2}$ and $I_{2,4}$ up to a sign ambiguity in the phase, just as for the case of three generations of quarks already discussed. The sign of the phase is fixed by the sign of $I_{4,4}^{(-)}$.

The multi-graded Hilbert series is

$$h(y,z) = \frac{1 + y^4 z^4}{(1-y^2)(1-y^4)(1-z^2)(1-z^4)(1-y^2z^2)(1-y^4z^2)}, \tag{6.9}$$

where y counts powers of m_E and z counts powers of m_5. The single variable Hilbert series is

$$H(q) = h(q,q) = \frac{1+q^8}{(1-q^2)^2(1-q^4)^3(1-q^6)}. \tag{6.10}$$

The q^8 term in the numerator shows that there is one degree-eight invariant $I_{4,4}^{(-)}$ which occurs, but that the square of this invariant is not independent and can be eliminated.

The number of denominator factors $p = 6$ is equal to the number of parameters, and $d_N = 8$, $d_D = 22$. The number of variables is $\dim V = 14$, since we have one 2×2 mass matrix, one 2×2 symmetric mass matrix, and their complex conjugates. Knop's inequality $14 \geq 22 - 8 \geq 6$ is satisfied, with an equality for the upper bound. The six parameters correspond to 2 charged lepton masses, 2 Majorana neutrino masses, one mixing angle and one phase.

The denominator of eq. (6.10) shows that there are two generators of degree two, three of degree four, and one of degree six, which agrees with the CP-even invariants in eq. (6.9). The numerator shows that there is an invariant of degree eight, whose square can be eliminated, which is $I_{4,4}^{(-)}$. The structure of the invariants for $n_g = 2$ is similar to that for quarks for $n_g = 3$.

Weak-basis invariants for two generations in the low-energy effective theory were studied previously by Branco, Lavoura and Rebelo [26]. They defined an invariant Q, related to $I_{4,4}^{(-)}$ by

$$2i\,\mathrm{Im}\,\mathrm{Tr}\,Q = I_{4,4}^{(-)}\,, \tag{6.11}$$

and showed that $Q = 0$ is a necessary and sufficient condition for CP conservation. This is consistent with our results, since the only CP-odd generating invariant is $I_{4,4}^{(-)}$.

6.2 The seesaw model

In this section, we analyze the lepton invariants in the seesaw theory for $n_g = n_g' = 2$ generations of fermions. There are three matrices in the lepton sector, m_ν, m_E and M, and their complex conjugates ${m_\nu}^\dagger$, ${m_E}^\dagger$ and $M^\dagger = M^*$.[7] From eq. (2.3), we see that only m_E transforms under $\mathcal{U}_{E^c}$, so it must always occur in the combination

$$X_E = {m_E}^\dagger m_E, \tag{6.12}$$

which transforms as

$$X_E \to \mathcal{U}_L^\dagger\; X_E\, \mathcal{U}_L \tag{6.13}$$

under the chiral flavor symmetry transformations. The mass matrices m_ν, $m_\nu^\dagger$, M and M^* transform as

$$\begin{aligned} m_\nu &\to {\mathcal{U}_{N^c}}^T\; m_\nu\, \mathcal{U}_L, \\ m_\nu^\dagger &\to {\mathcal{U}_L}^\dagger\; m_\nu^\dagger\, {\mathcal{U}_{N^c}}^*, \\ M &\to {\mathcal{U}_{N^c}}^T\; M\, \mathcal{U}_{N^c}, \\ M^* &\to {\mathcal{U}_{N^c}}^\dagger\; M^*\, {\mathcal{U}_{N^c}}^*\;. \end{aligned} \tag{6.14}$$

It is useful to define

$$\begin{aligned} X_\nu &\equiv m_\nu^\dagger m_\nu, \\ Z_\nu &= m_\nu m_\nu^\dagger, \\ {Z_\nu}^T &= {Z_\nu}^* = m_\nu^* {m_\nu}^T, \end{aligned} \tag{6.15}$$

which transform as

$$\begin{aligned} X_\nu &\to \mathcal{U}_L^\dagger\; X_\nu\, \mathcal{U}_L, \\ Z_\nu &\to {\mathcal{U}_{N^c}}^T\; Z_\nu\, {\mathcal{U}_{N^c}}^*, \\ {Z_\nu}^T &\to {\mathcal{U}_{N^c}}^\dagger\; {Z_\nu}^T\, \mathcal{U}_{N^c}, \end{aligned} \tag{6.16}$$

[7]It is worth emphasizing that in our notation m_ν refers to the Dirac mass matrix $m_\nu = Y_\nu v/\sqrt{2}$, not the Majorana mass matrix m_5 of the effective theory.

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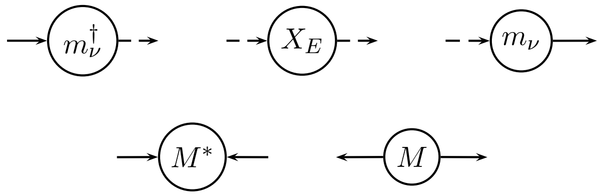

Figure 1. Graphical representation of the chiral transformation properties of the lepton mass matrices X_E, m_ν, $m_\nu^\dagger$, M and M^*. A solid line represents $\mathcal{U}_{N^c}$, and a dashed line $\mathcal{U}_L$. The invariants are obtained by forming graphs with no external lines.

as well as

$$\begin{aligned} X_N &\equiv M^* M, \\ Z_N &= M M^*, \\ Z_X &= m_\nu \, X_E \, m_\nu{}^\dagger \end{aligned} \tag{6.17}$$

which transform as

$$\begin{aligned} X_N &\to \mathcal{U}_{N^c}^\dagger \; X_N \, \mathcal{U}_{N^c}, \\ Z_N &\to \mathcal{U}_{N^c}{}^T \; Z_N \, \mathcal{U}_{N^c}{}^*, \\ Z_X &\to \mathcal{U}_{N^c}{}^T Z_X \, \mathcal{U}_{N^c}{}^*. \end{aligned} \tag{6.18}$$

Note that $Z_N{}^T = Z_N{}^* = X_N$.

The invariants involve three mass matrices, m_E, m_ν and M. One first can consider the simpler problem of studying invariants which only depend on two out of the three matrices. The first case, invariants involving only m_E and m_ν, consists of invariants formed from traces of X_E and X_ν only, with no insertions of M or M^*. These invariants are the same as the invariants in the quark sector with the substitutions $X_U \to X_\nu$ and $X_D \to X_E$. The second case, invariants involving only m_ν and M, are invariants which do not contain X_E. These have the same structure as invariants constructed in the low-energy theory, with the replacements $m_5 \to M$, $m_E \to m_\nu^T$, i.e. $X_E \to Z_\nu^T$.

The most general invariant involving all three matrices has the structure

$$\langle M^* A_1 M A_2^T \ldots M^* A_{2n-1} M A_{2n}^T \rangle \,, \tag{6.19}$$

where $A_i = \mathbb{1}$ or $A_i = m_\nu \mathcal{P}(X_E, X_\nu) m_\nu{}^\dagger$, where $\mathcal{P}$ is a polynomial in X_E and X_ν. This result can be obtained by representing the chiral transformations of the matrices graphically, as shown in figure 1. Products of matrices such as eq. (6.19) also occurred when studying rephasing invariants [5]. For rephasing invariants, one can factor long products into smaller ones, each involving at most four mixing matrices, using reconnection identities. This factorization is no longer possible for the case of mass-matrix invariants, which leads to an interesting and highly non-trivial structure for the invariants.

The basic invariants can be constructed using eq. (6.19) and eliminating higher powers of matrices by the Cayley-Hamilton identity eq. (5.5). The generators are:

$$\begin{aligned} I_{2,0,0} &= \langle X_E \rangle = \langle m_E{}^\dagger m_E \rangle \,, \\ I_{0,2,0} &= \langle X_\nu \rangle = \langle m_\nu{}^\dagger m_\nu \rangle \,, \end{aligned} \tag{6.20}$$

$$
\begin{aligned}
I_{0,0,2} &= \langle X_N \rangle = \langle M^* M \rangle \,, \qquad (6.21) \\
I_{4,0,0} &= \langle {X_E}^2 \rangle = \langle {m_E}^\dagger m_E {m_E}^\dagger m_E \rangle \,, \\
I_{2,2,0} &= \langle X_\nu X_E \rangle = \langle {m_\nu}^\dagger m_\nu {m_E}^\dagger m_E \rangle \,, \\
I_{0,4,0} &= \langle {X_\nu}^2 \rangle = \langle {m_\nu}^\dagger m_\nu {m_\nu}^\dagger m_\nu \rangle \,, \\
I_{0,2,2} &= \langle Z_\nu Z_N \rangle = \langle m_\nu {m_\nu}^\dagger M M^* \rangle \,, \\
I_{0,0,4} &= \langle {X_N}^2 \rangle = \langle M^* M M^* M \rangle \,, \\
I_{2,2,2} &= \langle Z_X Z_N \rangle = \langle m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M M^* \rangle \,, \\
I_{0,4,2} &= \langle M^* Z_\nu M {Z_\nu}^T \rangle = \langle M^* m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I_{2,4,2} &= \langle M^* Z_\nu M {Z_X}^T \rangle = \langle M^* m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_E}^T {m_E}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{2,4,2} &= \langle M^* Z_\nu Z_X M \rangle - \langle M^* Z_X Z_\nu M \rangle \\
&= \langle M^* m_\nu {m_\nu}^\dagger m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M \rangle - \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger m_\nu {m_\nu}^\dagger M \rangle \,, \\
I^{(-)}_{0,4,4} &= \langle Z_N Z_\nu M {Z_\nu}^T M^* \rangle - \langle M^* Z_\nu Z_N M {Z_\nu}^T \rangle \\
&= \langle M M^* m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T M^* \rangle - \langle M^* m_\nu {m_\nu}^\dagger M M^* M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I_{4,4,2} &= \langle M^* Z_X M {Z_X}^T \rangle \\
&= \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M {m_\nu}^* {m_E}^T {m_E}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{2,4,4} &= \langle Z_N Z_X M {Z_\nu}^T M^* \rangle - \langle M^* Z_X Z_N M {Z_\nu}^T \rangle \\
&= \langle M M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T M^* \rangle - \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M M^\dagger M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{2,6,2} &= \langle M^* Z_\nu Z_X M {Z_\nu}^T \rangle - \langle M^* Z_X Z_\nu M {Z_\nu}^T \rangle \\
&= \langle M^* m_\nu {m_\nu}^\dagger m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T \rangle - \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{4,4,4} &= \left\langle M^* Z_N Z_X M Z_X^T \right\rangle - \left\langle M^* Z_X^T Z_N M Z_X \right\rangle , \\
I^{(-)}_{4,6,2} &= \left\langle M^* Z_\nu Z_X M Z_X^T \right\rangle - \left\langle M^* Z_X Z_\nu M Z_X^T \right\rangle . \qquad (6.22)
\end{aligned}
$$

There are several invariants which can be immediately eliminated because they are polynomials in lower order invariants and which have not been listed above. These invariants include $I^{(+)}_{2,4,2}$, $I^{(+)}_{0,4,4}$, $I^{(+)}_{2,4,4}$, $I^{(+)}_{2,6,2}$, $I^{(+)}_{4,4,4}$ and $I^{(+)}_{4,6,2}$, which are related in an obvious way to the invariants in eq. (6.22) with superscripts $(-)$. The degree-eight invariants $I^{(+)}_{2,4,2}$ and $I^{(+)}_{0,4,4}$ are eliminated by the identities

$$
\begin{aligned}
0 &= I_{0,0,2} I^2_{0,2,0} I_{2,0,0} - I_{0,0,2} I_{0,4,0} I_{2,0,0} - 2 I_{0,2,0} I_{2,2,2} - 2 I_{0,2,2} I_{2,2,0} + 2 I^{(+)}_{2,4,2}, \\
0 &= I^2_{0,0,2} I^2_{0,2,0} - 2 I_{0,0,2} I_{0,4,2} - I_{0,0,4} I^2_{0,2,0} - 2 I^2_{0,2,2} + 2 I^{(+)}_{0,4,4}, \qquad (6.23)
\end{aligned}
$$

and the degree-ten invariants $I^{(+)}_{2,4,4}$ and $I^{(+)}_{2,6,2}$ are eliminated by the identities

$$
\begin{aligned}
0 &= I^2_{0,0,2} I_{0,2,0} I_{2,2,0} - 2 I_{0,0,2} I_{2,4,2} - I_{0,0,4} I_{0,2,0} I_{2,2,0} - 2 I_{0,2,2} I_{2,2,2} + 2 I^{(+)}_{2,4,4}, \\
0 &= I^2_{0,2,0} I_{0,2,2} I_{2,0,0} - 2 I_{0,2,0} I_{2,4,2} - I_{0,2,2} I_{0,4,0} I_{2,0,0} - 2 I_{0,4,2} I_{2,2,0} + 2 I^{(+)}_{2,6,2}. \qquad (6.24)
\end{aligned}
$$

The degree-twelve invariants $I^{(+)}_{4,4,4}$ and $I^{(+)}_{4,6,2}$ are also polynomials in lower order invariants, but we do not include the explicit identities here.

In eq. (6.22), there are three CP-even invariants of degree two, five of degree four, two of degree six, one of degree eight, and one of degree ten, for a grand total of 12 basic

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CP-even invariants. In addition, there are two CP-odd invariants of degree eight, two of degree ten and two of degree twelve, for a total of 6 basic CP-odd invariants. All of the invariants can be written as polynomials in these 18 basic invariants.

The multi-graded Hilbert series is

$$\begin{aligned} h(x,y,z) &= \frac{N}{D}, \\ N &= 1 + 2x^2y^4z^2 + y^4z^4 + x^2y^4z^4 + x^2y^6z^2 + x^4y^4z^4 + x^4y^6z^2 - x^2y^6z^6 - x^2y^8z^4 \\ &\quad - x^4y^6z^6 - x^4y^8z^4 - x^6y^8z^4 - 2x^4y^8z^6 - x^6y^{12}z^8, \\ D &= \left(1-x^2\right)\left(1-x^4\right)\left(1-y^2\right)\left(1-y^4\right)\left(1-z^2\right)\left(1-z^4\right)\left(1-x^2y^2\right)\left(1-y^2z^2\right) \\ &\quad \times \left(1-x^2y^2z^2\right)\left(1-y^4z^2\right)\left(1-x^4y^4z^2\right), \end{aligned} \tag{6.25}$$

where x, y, z count powers of m_E, m_ν and M, respectively. The Hilbert series $H(q) = h(q,q,q)$ is

$$H(q) = \frac{1 + q^6 + 3q^8 + 2q^{10} + 3q^{12} + q^{14} + q^{20}}{(1-q^2)^3(1-q^4)^5(1-q^6)(1-q^{10})}, \tag{6.26}$$

which has a palindromic numerator. The number of denominator factors $p = 10$ is equal to the number of parameters, and $d_N = 20$ and $d_D = 42$. The number of variables is $\dim V = 22$, because we have two 2×2 matrices with 4 independent entries, one 2×2 symmetric matrix with 3 independent entries, and their complex conjugates. Knop's inequality is $22 \geq 42 - 20 \geq 10$, and the upper bound is an equality. The 10 parameters in the lepton sector of the seesaw model for $n_g = n_g' = 2$ generations correspond to 2 charged lepton masses, 4 Majorana neutrino masses of the two light and the two heavy neutrinos, 2 angles and 2 phases.

One can see from the Hilbert series that the structure of invariants is far more complicated than in the quark case. The denominator factors $(1-q^2)^3(1-q^4)^5$ of eq. (6.26) corresponds to the generators $I_{2,0,0}$, $I_{0,2,0}$, $I_{0,0,2}$, $I_{4,0,0}$, $I_{2,2,0}$, $I_{0,4,0}$, $I_{0,2,2}$, $I_{0,0,4}$. At degree six, in addition to products of lower order invariants, there are two new invariants, $I_{2,2,2}$ and $I_{0,4,2}$. These two invariants correspond to the $(1-q^6)$ factor in the denominator, and the $+q^6$ term in the numerator. Since there is only one power of $(1-q^6)$ factor in the denominator, we know that there will be non-trivial relations involving the degree-six invariants. At degree eight, there are 3 new invariants from the $+3q^8$ term in the numerator in addition to products of lower degree invariants which make up the denominator. These are the three degree-eight invariants in eq. (6.22). There are three new invariants of degree twelve (from the $+3q^{12}$), but only two degree-twelve invariants in eq. (6.22). The third degree-twelve invariant is the square of the degree-six invariant corresponding to the $+q^6$ term in the numerator, so the square of this CP-even invariant cannot be removed. We have noted earlier that there must be non-trivial relations involving the degree-six invariants. These relations first occur at degree 14,

$$
\begin{aligned}
0 =\; & I_{0,0,2} I_{0,2,0} I^{(-)}_{2,6,2} + I^2_{0,2,0} I^{(-)}_{0,4,4} I_{2,0,0} - I^2_{0,2,0} I^{(-)}_{2,4,4} \\
& - I_{0,2,0} I_{0,2,2} I^{(-)}_{2,4,2} - I_{0,2,0} I^{(-)}_{0,4,4} I_{2,2,0} - 2 I_{0,2,2} I^{(-)}_{2,6,2} \\
& - I_{0,4,0} I^{(-)}_{0,4,4} I_{2,0,0} + 2 I_{0,4,0} I^{(-)}_{2,4,4} + 2 I_{0,4,2} I^{(-)}_{2,4,2} \\
0 =\; & I^2_{0,0,2} I_{0,2,0} I^{(-)}_{2,4,2} - I^2_{0,0,2} I^{(-)}_{2,6,2} + I_{0,0,2} I_{0,2,0} I^{(-)}_{2,4,4} \\
& - I_{0,0,2} I_{0,2,2} I^{(-)}_{2,4,2} - I_{0,0,2} I_{2,2,0} I^{(-)}_{0,4,4} - I_{0,0,4} I_{0,2,0} I^{(-)}_{2,4,2} \\
& + 2 I_{0,0,4} I^{(-)}_{2,6,2} - 2 I_{0,2,2} I^{(-)}_{2,4,4} + 2 I_{2,2,2} I^{(-)}_{0,4,4},
\end{aligned}
\tag{6.27}
$$

and are non-linear relations involving the two degree-six invariants. One can proceed to higher degrees — there are six relations of degree 16, etc., and verify the number of independent invariants at each degree agrees with eq. (6.26). The details of the relations are not important. The main purpose of giving eq. (6.27) is to show that there can be non-linear relations among the generating invariants. To completely unravel all of the non-linear relations requires going beyond degree 20, the highest power of q in the numerator of eq. (6.26).

7 Lepton invariants for three generations

In this section, we consider the lepton invariants in the low-energy and high-energy theories for three generations of fermions. The number of invariants is far greater than for two generations, and there are many relations between them. For the low-energy theory, we give the Hilbert series, and the invariants which correspond to the denominator factors. For three generations, even the Hilbert series proved too difficult to compute. For this case, we make some general remarks, and discuss some invariants considered previously by Branco et al. [26, 27], and by Davidson and Kitano [15].

7.1 The standard model effective theory

The invariants involving only X_E are $I_{2,0} = \langle X_E \rangle$, $I_{4,0} = \langle {X_E}^2 \rangle$ and $I_{6,0} = \langle {X_E}^3 \rangle$, whereas the invariants involving only m_5 and ${m_5}^*$ are $I_{0,2} = \langle X_5 \rangle$, $I_{0,4} = \langle {X_5}^2 \rangle$ and $I_{0,6} = \langle {X_5}^3 \rangle$.

The invariants involving X_E, m_5 and ${m_5}^*$ are of the form

$$
\langle {m_5}^* \; ({X_E}^{r_1})^T \; m_5 \; {X_E}^{s_1} \; \ldots {m_5}^* \; ({X_E}^{r_n})^T \; m_5 \; {X_E}^{s_n} \rangle \tag{7.1}
$$

for integers r_i and s_i. The Cayley-Hamilton theorem implies that all powers r_i and s_i greater than two in eq. (7.1) can be rewritten in terms of lower order invariants. Thus, one needs to consider traces of matrix products containing the matrices X_E, X_5, $\left({m_5}^* \; {X_E}^T \; m_5\right)$, and $\left({m_5}^* \; \left({X_E}^2\right)^T \; m_5\right)$ at most twice. Identity eq. (5.16) cannot be used to eliminate traces with multiple powers of m_5, because $\langle m_5 A m_5 B \rangle$ gets converted to traces of the form $\left\langle m_5^2 AB \right\rangle$ which are no longer invariant. There are many basic invariants, which involve a single trace, up to degree $m_5^{10} m_E^{12}$, and we do not list them all here. The ones up to degree twelve, which are sufficient for the denominator of the Hilbert series (and hence to determine the parameters) are:

$$\begin{aligned}
I_{2,0} &= \langle X_E\rangle = \langle {m_E}^\dagger m_E\rangle\,,\\
I_{0,2} &= \langle X_5\rangle = \langle {m_5}^* m_5\rangle\,,\\
I_{4,0} &= \langle {X_E}^2\rangle = \langle \left({m_E}^\dagger m_E\right)^2\rangle\,,\\
I_{2,2} &= \langle X_E X_5\rangle = \langle {m_E}^\dagger m_E {m_5}^* m_5\rangle\,,\\
I_{0,4} &= \langle {X_5}^2\rangle = \langle ({m_5}^* m_5)^2\rangle\,,\\
I_{6,0} &= \langle {X_E}^3\rangle = \langle \left({m_E}^\dagger m_E\right)^3\rangle\,,\\
I'_{4,2} &= \langle {X_E}^2 X_5\rangle = \langle \left({m_E}^\dagger m_E\right)^2 {m_5}^* m_5\rangle\,,\\
I_{4,2} &= \langle {m_5}^*\ {X_E}^T\ m_5\ X_E\rangle = \langle {m_5}^*\ {m_E}^T {m_E}^*\ m_5\ {m_E}^\dagger m_E\rangle\,,\\
I_{2,4} &= \langle X_E {X_5}^2\rangle = \langle {m_E}^\dagger m_E\,({m_5}^* m_5)^2\rangle\,,\\
I_{0,6} &= \langle {X_5}^3\rangle = \langle ({m_5}^* m_5)^3\rangle\,,\\
I_{6,2} &= \langle {m_5}^*\ {X_E}^T\ m_5\ {X_E}^2\rangle = \langle {m_5}^*\ {m_E}^T {m_E}^*\ m_5\ \left({m_E}^\dagger m_E\right)^2\rangle\,,\\
I^{(\pm)}_{4,4} &= \langle {m_5}^*\ {X_E}^T\ m_5\ {m_5}^*\ m_5\ X_E\rangle \pm \langle {m_5}^*\ m_5\ {m_5}^*\ {X_E}^T\ m_5\ X_E\rangle\\
&= \langle {m_5}^*\ {m_E}^T {m_E}^*\ m_5\ {m_5}^*\ m_5\ {m_E}^\dagger m_E\rangle \pm \langle {m_5}^*\ m_5\ {m_5}^*\ {m_E}^T {m_E}^*\ m_5\ {m_E}^\dagger m_E\rangle\,,\\
I_{8,2} &= \langle {m_5}^*\ ({X_E}^T)^2\ m_5\ {X_E}^2\rangle = \langle {m_5}^*\ \left({m_E}^T {m_E}^*\right)^2\ m_5\ \left({m_E}^\dagger m_E\right)^2\rangle\,,\\
I^{(\pm)}_{6,4} &= \langle {m_5}^*\ {X_E}^T\ m_5\ {m_5}^*\ m_5\ {X_E}^2\rangle \pm \langle {m_5}^*\ m_5\ {m_5}^*\ {X_E}^T\ m_5\ {X_E}^2\ \rangle\\
&= \langle {m_5}^*\ {m_E}^T {m_E}^*\ m_5\ {m_5}^*\ m_5\ \left({m_E}^\dagger m_E\right)^2\rangle \pm \langle {m_5}^*\ m_5\ {m_5}^*\ {m_E}^T {m_E}^*\ m_5\ \left({m_E}^\dagger m_E\right)^2\rangle\,,\\
I^{(\pm)}_{8,4} &= \langle {m_5}^*\ ({X_E}^T)^2\ m_5\ {m_5}^*\ m_5\ {X_E}^2\ \rangle \pm \langle {m_5}^*\ m_5\ {m_5}^*\ ({X_E}^T)^2 m_5\ {X_E}^2\rangle\\
&= \langle {m_5}^*\ \left({m_E}^T {m_E}^*\right)^2\ m_5\ {m_5}^*\ m_5\ \left({m_E}^\dagger m_E\right)^2\rangle\\
&\quad \pm \langle {m_5}^*\ m_5\ {m_5}^*\ \left({m_E}^T {m_E}^*\right)^2 m_5\ \left({m_E}^\dagger m_E\right)^2\rangle\,.
\end{aligned} \tag{7.2}$$

The multi-graded Hilbert series is

$$\begin{aligned}
h(y,z) &= \frac{N}{D},\\
N &= -y^{24}z^{18} - 2y^{20}z^{14} - 2y^{20}z^{12} - y^{20}z^{10} - 2y^{18}z^{14} - 3y^{18}z^{12} - y^{18}z^{10} - 3y^{16}z^{14}\\
&\quad -3y^{16}z^{12} - 3y^{16}z^{10} - y^{16}z^8 - y^{16}z^6 - y^{14}z^{14} - y^{14}z^{12} - y^{14}z^{10} - 2y^{14}z^8\\
&\quad -y^{14}z^6 - y^{12}z^{14} + y^{12}z^4 + y^{10}z^{12} + 2y^{10}z^{10} + y^{10}z^8 + y^{10}z^6 + y^{10}z^4 + y^8z^{12}\\
&\quad +y^8z^{10} + 3y^8z^8 + 3y^8z^6 + 3y^8z^4 + y^6z^8 + 3y^6z^6 + 2y^6z^4 + y^4z^8 + 2y^4z^6\\
&\quad +2y^4z^4 + 1,\\
D &= \left(1-y^2\right)\left(1-y^4\right)\left(1-y^6\right)\left(1-z^2\right)\left(1-z^4\right)\left(1-z^6\right)\left(1-y^2z^2\right)\left(1-y^4z^2\right)^2\\
&\quad \times\left(1-y^2z^4\right)\left(1-y^6z^2\right)\left(1-y^4z^4\right)\left(1-y^8z^2\right),
\end{aligned} \tag{7.3}$$

where y counts powers of m_E and z counts powers of m_5. The single-variable series $H(q) = h(q,q)$ is

$$H(q)= \frac{1+q^6+2q^8+4q^{10}+8q^{12}+7q^{14}+9q^{16}+10q^{18}+9q^{20}+7q^{22}+8q^{24}+4q^{26}+2q^{28}+q^{30}+q^{36}}{(1-q^2)^2\,(1-q^4)^3\,(1-q^6)^4\,(1-q^8)^2\,(1-q^{10})}. \tag{7.4}$$

The number of denominator factors $p = 12$ is equal to the number of parameters, and $d_N = 36$ and $d_D = 66$. The number of variables is $\dim V = 30$, because we have one 3×3 matrix with 9 independent entries, one 3×3 symmetric matrix with 6 independent entries, and their complex conjugates. Knop's inequality is $30 \geq 66 - 36 \geq 12$, and the upper bound is an equality. Note that the numerator is palindromic. The 12 parameters consist of 3 charged lepton masses, 3 Majorana light neutrino masses, 3 angles and 3 phases.

The Hilbert series eq. (7.4) has a complicated numerator, which shows that the structure of the invariant ring is highly non-trivial. From the denominator of eq. (7.4), we see that there are two generators of degree two, three of degree four, four of degree six, two of degree eight, and one of degree 10, which can be multiplied freely, with no relations. These account for most of the invariants in eq. (7.2), but there remains one CP-even invariant each of degrees 6, 10, 12, and one CP-odd invariant each of degrees 8, 10, 12. These contribute $q^6 + q^8 + 2q^{10} + 2q^{12}$ to the numerator in eq. (7.4). The coefficient of q^8 in the numerator of eq. (7.4) is 2. Where does the other degree-eight invariant not in eq. (7.2) come from? The degree-six invariant that corresponds to the numerator factor q^6 can be multiplied by either of the two degree invariants, $I_{2,0}$ or $I_{0,2}$, to give two additional degree-8 invariants. One of these can be written as a polynomial in lower order invariants; the other survives. One can continue this analysis to arbitrarily high order — the entire invariant structure is encoded in a very compact way in the Hilbert series eq. (7.4). An explicit example of the construction just discussed is given in section 6.2 for the high-energy theory with $n_g = 2$, which provides a simpler example of an invariant ring with non-trivial relations.

For three generations, Branco, Lavoura and Rebelo [26] defined four invariants:

$$\begin{aligned}
2iI_1 &= I^{(-)}_{4,4} \\
2iI_2 &= \left\langle X_E m_5^* m_5 m_5^* m_5 m_5^* X_E^T m_5 \right\rangle - \text{c.c.} \\
2iI_3 &= \left\langle X_E m_5^* m_5 m_5^* m_5 m_5^* X_E^T m_5 m_5^* m_5 \right\rangle - \text{c.c.} \\
2iI_4 &= \det\left[m_5 X_E m_5^* + m_5^* X_E^T m_5 \right] - \text{c.c.}
\end{aligned} \tag{7.5}$$

of degrees $(4,4)$, $(4,6)$, $(4,8)$ and $(6,6)$, and showed that the vanishing of these invariants implies CP conservation. The CP-violating invariants of eq. (7.2) correspond to the denominator factors of the Hilbert series. There are additional CP-violating invariants not listed which correspond to terms in the numerator.

7.2 The seesaw model

The invariants involve three mass matrices, m_E, m_ν and M. One first can consider the simpler problem of studying invariants which only depend on two out of the three matrices. The first case, invariants involving only m_E and m_ν, consists of invariants formed from traces of X_E and X_ν only, with no insertions of M or M^*. These invariants are in direct analogy to the invariants of the quark sector with the substitutions $X_U \to X_\nu$ and $X_D \to X_E$. The second case, invariants involving only m_ν and M, are invariants which do not contain X_E. These have the same structure as invariants constructed in the low-energy theory, with the replacements $m_5 \to M$, $m_E \to m_\nu^T$, i.e. $X_E \to Z_\nu^T$.

The most general invariant involving all three matrices has the structure

$$\langle M^* A_1 M A_2^T \dots M^* A_{2n-1} M A_{2n}^T \rangle \,, \tag{7.6}$$

where $A_i = \mathbb{1}$ or $A_i = m_\nu \mathcal{P}(X_E, X_\nu) {m_\nu}^\dagger$, where $\mathcal{P}$ is a polynomial in X_E and X_ν. The generating invariants are given by using eq. (7.6). In this case, there are a very large number of generating invariants. They include all those discussed earlier in the seesaw theory for two generations, as well as many other.

For $n_g = n_g' = 3$ generations, there are 21 parameters which consist of 9 masses, 6 angles and 6 phases. The 9 masses are the 3 charged lepton masses, 3 light Majorana neutrino masses and 3 heavy Majorana neutrino masses. There are 3 angles in the mixing matrix V and 3 angles in the mixing matrix W. There is one δ-type phase in V and in W, two Majorana phases Ψ' in W, and 2 phases $\bar{\Phi}$ which are not removeable when V and W are considered together.

We have been unable to construct the multi-graded and one-variable Hilbert series in this case. However, it is clear that the structure of the invariant relations is extremely complicated. There are a number of constraints on the form of the one-variable Hilbert series. The denominator must be a product of $p = 21$ factors. The numerator must be palindromic, and d_N and d_D must satisfy the Knop inequality $48 \geq d_D - d_N \geq 21$ since $\dim V = 48$. The number of variables $\dim V = 48$ results because there are two 3×3 matrices m_E and m_ν with 9 independent entries each, one 3×3 symmetric matrix M with 6 independent entries, and the complex conjugates of the three matrices.

Ref. [27] defined six invariants in the seesaw theory,

$$\begin{aligned}
2iI_1 &= \left\langle Y_\nu Y_\nu^\dagger M^* M M^* (Y_\nu Y_\nu^\dagger)^T M \right\rangle - \text{c.c.} \\
2iI_2 &= \left\langle Y_\nu Y_\nu^\dagger M^* M M^* M M^* (Y_\nu Y_\nu^\dagger)^T M \right\rangle - \text{c.c.} \\
2iI_3 &= \left\langle Y_\nu Y_\nu^\dagger M^* M M^* M M^* (Y_\nu Y_\nu^\dagger)^T M M^* M \right\rangle - \text{c.c.}
\end{aligned} \tag{7.7}$$

which involve CP-violating phases which are relevant for leptogenesis, as well as

$$\begin{aligned}
2i\tilde{I}_1 &= \left\langle Y_\nu X_E Y_\nu^\dagger M^* M M^* (Y_\nu X_E Y_\nu^\dagger)^T M \right\rangle - \text{c.c.} \\
2i\tilde{I}_2 &= \left\langle Y_\nu X_E Y_\nu^\dagger M^* M M^* M M^* (Y_\nu X_E Y_\nu^\dagger)^T M \right\rangle - \text{c.c.} \\
2i\tilde{I}_3 &= \left\langle Y_\nu X_E Y_\nu^\dagger M^* M M^* M M^* (Y_\nu X_E Y_\nu^\dagger)^T M M^* M \right\rangle - \text{c.c.}
\end{aligned} \tag{7.8}$$

which involve the other phases.

Ref. [15] defines an invariant

$$2iI_1 = \left\langle \kappa^\dagger \kappa \kappa^\dagger (Y_\nu^T Y_\nu^*)^{-1} \kappa (Y_\nu^\dagger Y_\nu)^{-1} \right\rangle \tag{7.9}$$

for leptogenesis, where κ is m_5 with factors of the Higgs vacuum expectation value removed. This is not a polynomial in the basic variables of the seesaw model. It can be related to the invariants considered here using the formulæ given below.

Invariants in the seesaw model can be related to those of the low-energy effective theory. The basic relation is eq. (2.11), which relates the neutrino mass matrices in the seesaw model to the Majorana mass matrix m_5 in the low-energy effective theory. Clearly, the relations between the invariants cannot be polynomial, since inverse powers of M are involved, but one can write the low-energy invariants in terms of a rational function of the high-energy invariants. The basic identities are:

$$\begin{aligned} \det A \; A^{-1} &= \langle A \rangle - A \\ \det A &= \frac{1}{2} \langle A \rangle^2 - \frac{1}{2} \langle A^2 \rangle \end{aligned} \tag{7.10}$$

for 2×2 matrices, and

$$\begin{aligned} \det A \; A^{-1} &= A^2 - A \langle A \rangle - \frac{1}{2} \langle A^2 \rangle + \frac{1}{2} \langle A \rangle^2 \\ \det A &= \frac{1}{3} \langle A^3 \rangle - \frac{1}{2} \langle A^2 \rangle \langle A \rangle + \frac{1}{6} \langle A \rangle^3 \end{aligned} \tag{7.11}$$

for 3×3 matrices, which can be combined with

$$C_5 = Y_\nu^T M^{-1} Y_\nu = Y_\nu^T (M^* M)^{-1} M^* Y_\nu \tag{7.12}$$

to obtain the desired relations using $A = M^* M$, and substituting for C_5 (i.e. m_5) in the expressions for the low-energy invariants. The expressions are valid as long as $\det M^* M \neq 0$, i.e. as long as the singlet neutrinos are heavy and the transition to a low-energy effective theory is valid.

8 Conclusions

We have used the mathematics of invariant theory to classify the independent invariants of the Standard Model effective theory and its high-energy seesaw model and to study the non-trivial structure of relations (syzygies) among the invariant generators. The complete classification of invariants and the Hilbert series have been obtained for the Standard Model effective theory with a dimension-five Majorana neutrino mass operator. A complete solution also has been obtained for the renormalizable seesaw model with $n_g = n_g' = 2$ fermion generations. The lepton sector of the seesaw model involves three different mass matrices, the charged lepton mass matrix, the Dirac mass matrix of the weakly-interacting doublet neutrinos and the Majorana mass matrix of the gauge-singlet neutrinos. The invariant structure is very complicated. In the case of $n_g = n_g' = 3$ generations of fermions, we have been unable to find the Hilbert series for the invariant generators, and thus the structure of the syzygy relations for three generations remains an open problem.

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